

On convergence of solutions to equilibria for fully nonlinear parabolic problems with nonlinear boundary conditions

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Abstract

We show convergence of solutions to equilibria for fully nonlinear parabolic evolution systems with nonlinear boundary conditions in situations where the set of equilibria is non-discrete, but forms a finite-dimensional C^2 -manifold which is normally stable. We apply the parabolic Hölder setting for such problems which allows to deal with nonlocal terms. As an illustration of the scope of our result we show that the lens-shaped networks generated by circular arcs are stable under the surface diffusion flow.

Keywords. nonlinear stability, fully nonlinear parabolic systems, nonlinear boundary conditions, nonlocal PDE, normally stable, free boundary problems, surface diffusion flow, triple junctions, lens-shaped network

AMS subject classifications: 35K55, 35B35, 37L15, 53C44

1 Introduction

We study convergence of solutions toward equilibria for fully nonlinear parabolic evolution systems with nonlinear boundary conditions in the parabolic Hölder setting, in situations where the equilibria forms a finite dimensional manifold. Our main result can be summarized as follows: suppose that for a fully nonlinear parabolic evolution system with nonlinear boundary conditions we have a C^2 -manifold of equilibria \mathcal{E} such that at a point $u_* \in \mathcal{E}$, the kernel $N(A_0)$ of the linearization A_0 is isomorphic to the tangent space \mathcal{E} at u_* , the eigenvalue 0 of A_0 is semi-simple, and the remaining spectral part of the linearization A_0 is stable. Then solutions starting nearby u_* satisfying some compatibility conditions exist globally and converge to some point on \mathcal{E} .

This situation described above occurs frequently in applications and it is called the *generalized principle of linearized stability*, and the equilibrium u_* is then termed *normally stable*.

A typical example for this situation to occur is the case where the equations under consideration involve symmetries, i.e. are invariant under the action of a

Lie group \mathcal{G} . If then u_* is an equilibrium, the manifold \mathcal{E} includes the action of \mathcal{G} on u_* and the manifold $\mathcal{G}u_*$ is a subset of \mathcal{E} . For instance, this is true in a case of the surface diffusion flow as the set of equilibria is typically invariant under translation and under dilation (at least locally).

A standard method to handle situations as described above is to refer to *center manifold theory*. However, the theory of center manifolds is a technically difficult matter. Therefore it seems desirable to have a simpler, direct approach to the generalized principle of linearized stability which avoids the technicalities of center manifold theory, in particular in situations with highly nonlinear boundary conditions, where the center manifold theory is difficult to apply.

Such an approach has been introduced by Prüss, Simonett and Zacher [22] for abstract quasilinear problems and also for vector-valued quasilinear parabolic systems with vector-valued nonlinear boundary conditions in the framework of L_p -maximal regularity. This approach is extended in [23] to cover a broader setting and a broader class of nonlinear parabolic equations, including fully nonlinear equations but just for abstract problems, i.e. without nonlinear boundary conditions.

However, in application for instance in geometric evolution problems at the presence of triple junctions the corresponding systems involve nonlocal terms (due to the movement of triple junctions, see [15, 10]) and nonlinear boundary conditions which can not be treated by the L^p setting introduced in [22] (because of nonlocal terms involving highest spatial derivatives) or by the general setting introduced in [23] (because of nonlinear boundary conditions).

The purpose of this paper is to extend the approaches given in [22, 23] to cover fully nonlinear problems with nonlinear boundary conditions in the parabolic Hölder spaces. Within the parabolic Hölder setting we are allowed to deal with those nonlocal terms. In a forthcoming paper within this setting we apply our results and we are able to prove that the stationary solutions of the form of the "Mercedes star" and also the stationary solutions of the form of the 2-D double bubble are stable under the surface diffusion flow.

The paper is organised as follows. In Section 2 we formulate the problem and in Section 3 we state and prove our main result, i.e. Theorem 3.1. The proof relies on results for the asymptotic behaviour of linear inhomogeneous systems which are given in the appendix. In that direction, extending the result stated in [20], we construct explicitly an extension operator for the case of vector-valued unknowns (see Subsection 5.2.1).

Section 4 illustrates the scope of our main result, as we show that the lens-shaped networks generated by circular arcs are stable under the surface diffusion flow. Indeed the lens-shaped networks are the simplest examples of the more general triple junctions where the resulting PDE has nonlocal terms in the highest order derivatives. Therefore we work in function spaces which yield classical solutions.

2 Fully nonlinear parabolic problems with nonlinear boundary conditions in parabolic Hölder setting

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with boundary $\partial\Omega \in C^{2m+\alpha}$ with $0 < \alpha < 1$. The outer normal at a point $x \in \partial\Omega$ will be denoted by $\nu(x)$. We consider the following problem

$$\begin{cases} \partial_t u(t, x) + A(u(t, \cdot))(x) = F(u(t, \cdot))(x), & x \in \overline{\Omega}, \quad t > 0, \\ B_j(u(t, \cdot))(x) = G_j(u(t, \cdot))(x), & x \in \partial\Omega, \quad j = 1, \dots, mN, \\ u(0, x) = u_0(x), & x \in \overline{\Omega}. \end{cases} \quad (2.1)$$

Here $u : \overline{\Omega} \times [0, \infty) \rightarrow \mathbb{R}^N$ and A is a linear $2m$ order differential operator of the form

$$(Au)(x) = \sum_{|\gamma| \leq 2m} a_\gamma(x) \nabla^\gamma u(x), \quad x \in \overline{\Omega},$$

and the B_j 's are linear differential operators of order m_j ,

$$(B_j u)(x) = \sum_{|\beta| \leq m_j} b_\beta^j(x) \nabla^\beta u(x), \quad x \in \partial\Omega, \quad j = 1, \dots, mN.$$

Here the coefficients a_γ are $N \times N$ square matrices, b_β^j are N -dimensional row-vectors and

$$0 \leq m_1 \leq m_2 \leq \dots \leq m_{mN} \leq 2m - 1.$$

Let $n_j \geq 0$ be the number of j th order boundary conditions for $j = 0, \dots, 2m - 1$.

The assumptions on F and G_j are the following.

(R_1) $F : B(0, R) \subset C^{2m}(\overline{\Omega}) \rightarrow C(\overline{\Omega})$ is C^1 with Lipschitz continuous derivative, $F(0) = 0$, $F'(0) = 0$, and the restriction of F to $B(0, R) \subset C^{2m+\alpha}(\overline{\Omega})$ has values in $C^\alpha(\overline{\Omega})$ and is continuously differentiable.

$G_j : B(0, R) \subset C^{m_j}(\overline{\Omega}) \rightarrow C(\partial\Omega)$ is C^2 with Lipschitz continuous second order derivative, $G_j(0) = 0$, $G_j'(0) = 0$, and the restriction of G_j to $B(0, R) \subset C^{2m+\alpha}(\overline{\Omega})$ has values in $C^{2m+\alpha-m_j}(\partial\Omega)$ and is continuously differentiable.

The coefficient are subject to the following regularity assumptions.

(R_2) The elements of the matrix $a_\gamma(x)$ belong to $C^\alpha(\overline{\Omega})$,

The elements of the matrix $b_\beta^j(x)$ belong to $C^{2m+\alpha-m_j}(\partial\Omega)$.

We set $B = (B_1, \dots, B_{mN})$ and $G = (G_1, \dots, G_{mN})$.

Condition (R_1) is understood in the sense that each component of the indicated vectors belongs to the corresponding spaces.

Remark 2.1. Reading the assumption (R_1) we see that this kind of nonlinearity include a very general class of perturbations, for instance, it allows the following nonlocality: F can depend on $D^\alpha u(x_0)$ where x_0 is a point on $\overline{\Omega}$ with $|\alpha| = 2m$.

We will employ the following setting for this problem:

$$X = C(\overline{\Omega}), \quad X_0 = C^\alpha(\overline{\Omega}), \quad X_1 = C^{2m+\alpha}(\overline{\Omega})$$

Note that $X_1 \hookrightarrow X_0 \hookrightarrow X$. We denote the norm in X_j by $|\cdot|_j$ for $j = 0, 1$ and the norm in X by $|\cdot|$. Moreover, for any normed space Y , $B_Y(u, r)$ denotes the open ball in Y with radius $r > 0$ around $u \in Y$.

Let $\mathcal{E} \subset X_1$ denote the set of equilibrium solutions of (2.1), which means that

$$u \in \mathcal{E} \iff u \in X_1, \quad Au = F(u) \quad \text{in } \Omega \quad \text{and} \quad Bu = G(u) \quad \text{on } \partial\Omega. \quad (2.2)$$

It follows from assumption (R_1) that $u_* \equiv 0$ belongs to \mathcal{E} . Although u_* is zero, we will often write u_* instead of 0 to emphasize that we deal with an equilibrium.

We assume that u_* is contained in a k -dimensional manifold of equilibria. This means that there is an open subset $U \subset \mathbb{R}^k$, $0 \in U$, and a C^2 -function $\Psi : U \rightarrow X_1$, such that

- $\Psi(U) \subset \mathcal{E}$ and $\Psi(0) = u_* \equiv 0$,
- the rank of $\Psi'(0)$ equals k ,
- $A\Psi(\zeta) = F(\Psi(\zeta)) \quad \text{in } \Omega, \quad \text{for all } \zeta \in U,$ (2.3)
- $B\Psi(\zeta) = G(\Psi(\zeta)) \quad \text{on } \partial\Omega, \quad \text{for all } \zeta \in U,$ (2.4)

We assume further that there are no other equilibria near u_* in X_1 , i.e. for some $r_1 > 0$,

$$\mathcal{E} \cap B_{X_1}(u_*, r_1) = \Psi(U).$$

The linearization of (2.1) at u_* is given by the operator A_0 which is the realization of A with homogeneous boundary conditions in $X = C(\overline{\Omega})$, i.e., the operator with domain

$$D(A_0) = \left\{ u \in C(\overline{\Omega}) \cap \bigcap_{1 < p < +\infty} W^{2m,p}(\Omega) : Au \in X, \quad Bu = 0 \text{ on } \partial\Omega \right\}, \quad (2.5)$$

$$A_0 u = Au, \quad u \in D(A_0),$$

where we used the fact that $F'(0) = G'(0) = 0$. By assumption (R_2) we obtain

$$A_0|_{C^{2m+\alpha}(\overline{\Omega})} : C^{2m+\alpha}(\overline{\Omega})|_{N(B)} \rightarrow C^\alpha(\overline{\Omega}).$$

Remark 2.2. Since Ω is bounded, the domain of A_0 is compactly embedded into $C(\overline{\Omega})$, the resolvent operators $(\lambda I - A_0)^{-1}$ are compact, and the spectrum consists of a sequence of isolated eigenvalues.

Next we consider the property of maximal regularity in the parabolic Hölder spaces for the pair (A, B) , and in particular for the operator A_0 . For this we only need to consider the principal parts of the corresponding differential operator and of the boundary operators, i.e.

$$A_*(x, D) = \sum_{|\gamma|=2m} i^{2m} a_\gamma(x) D^\gamma,$$

$$B_{j*}(x, D) = \sum_{|\beta|=m_j} i^{m_j} b_{\beta}^j(x) D^\beta,$$

for $j = 1, \dots, mN$. Note that we use the notation $D = -i\nabla$, hence $\nabla^\beta = i^{|\beta|} D^\beta$. Based on the results of V.A. Solonnikov [26], strong parabolicity of A_* and the Lopatinskii-Shapiro condition for (A_*, B_*) are sufficient for Hölder-maximal regularity of A_0 , see Theorem VI.21 in [11]. These conditions read as follows:

(SP) A is strongly parabolic: For all $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^n$, $|\xi| = 1$,

$$\sigma(A_*(x, \xi)) \subset \mathbb{C}_+.$$

(LS) (Lopatinskii-Shapiro condition) For all $x \in \partial\Omega$, $\xi \in \mathbb{R}^n$, with $\xi \cdot \nu(x) = 0$, $\lambda \in \overline{\mathbb{C}_+}$, $\lambda \neq 0$, and $h \in \mathbb{C}^{mN}$, the system of ordinary differential equations on the half-line

$$\begin{aligned} \lambda v(y) + A_*(x, \xi + i\nu(x)\partial_y)v(y) &= 0, & y > 0, \\ B_{j*}(x, \xi + i\nu(x)\partial_y)v(0) &= h_j, & j = 1, \dots, mN, \end{aligned}$$

admits a unique solution $v \in C_0(\mathbb{R}_+; \mathbb{C}^N)$,

where $C_0(\mathbb{R}_+; \mathbb{C}^N)$ is the space of continuous functions which vanish at infinity.

Remark 2.3. *The uniformly strongly parabolic condition i.e. (SP) implies the root condition (cf. Amann [5, Lemma 6.1] or Morrey [21, P. 255]). Concerning the Complementing Condition (LS), here it is formulated in a non-algebraic way but one can find the equivalence of this formulation to the algebraic formulation in Eidelman and Zhitarashu [11, Chap. I.2]. See also Lemma 6.2 in [5].*

We shall also use known results on generators of analytic semigroups, the characterization of interpolation spaces and Hölder regularity for elliptic systems, which we collect in the following theorem.

Theorem 2.4. *Suppose the conditions (R_2) , (SP) and (LS) hold. Then*

- (i) *The operator $-A_0$ is sectorial.*
- (ii) *For each $\theta \in (0, 1)$ such that $2m\theta \notin \mathbb{N}$, we have*

$$D_{-A_0}(\theta, \infty) = \{\varphi \in C^{2m\theta}(\overline{\Omega}) : B_j \varphi = 0 \text{ if } m_j \leq [2m\theta]\}$$

and the $C^{2m\theta}$ -norm is equivalent to the $D_{-A_0}(\theta, \infty)$ -norm.

(iii) For each $k = 1, \dots, 2m - 1$ we have

$$C_{\mathcal{B}}^k(\overline{\Omega}) := \{\varphi \in C^k(\overline{\Omega}) : B_j \varphi = 0 \text{ if } m_j < k\} \hookrightarrow D_{-A_0}(\frac{k}{2m}, \infty)$$

where $C_{\mathcal{B}}^k(\overline{\Omega})$ is given the norm of $C^k(\overline{\Omega})$.

(iv)

$$\left\{ \varphi \in \bigcap_{p>1} W^{2m,p}(\Omega) : A\varphi \in C^\alpha(\overline{\Omega}), \quad B_j \varphi \in C^{2m+\alpha-m_j}(\partial\Omega), \right. \\ \left. j = 1, \dots, mN \right\} \subset C^{2m+\alpha}(\overline{\Omega})$$

and there exist a constant C such that

$$\|\varphi\|_{C^{2m+\alpha}(\overline{\Omega})} \leq C \left(\|A\varphi\|_{C^\alpha(\overline{\Omega})} + \|\varphi\|_{C(\overline{\Omega})} + \sum_{j=1}^{mN} \|B_j \varphi\|_{C^{2m+\alpha-m_j}(\partial\Omega)} \right). \quad (2.6)$$

Proof. Concerning (i) and (ii), see [2, Remark 5.1]. (iii) follows from the characterization of $D_{-A_0}(\frac{k}{2m}, \infty)$ provided in [1], see precisely Remark 5.1 in [1]. Concerning (iv), one must observe that the results of [3] prove the estimate (2.6), whereas the inclusion in $C^{2m+\alpha}(\overline{\Omega})$ is a consequence of the existence theorems proved in [17, Section 5]. \square

Note that in the case of a single elliptic equation for one unknown function, the same results are proven in [20, Theorem 5.2].

Differentiating (2.3) and (2.4) w.r.t. ζ we obtain for $\zeta = 0$

$$\begin{cases} A\Psi'(0) = 0 & \text{in } \Omega, \\ B\Psi'(0) = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.7)$$

This shows that the image of $\Psi'(0)$ is contained in the kernel $N(A_0)$ of A_0 , and also that $T_{u_*}(\mathcal{E})$, the tangential space of \mathcal{E} at u_* , is contained in $N(A_0)$.

We need to put more assumption on the coefficient b_j^β , which is called ‘normality condition’ and will be used in the construction of the extension operator presented in the appendix.

$$\left\{ \begin{array}{l} \text{for each } x \in \partial\Omega, \text{ the matrix } \begin{pmatrix} \sum_{|\beta|=k} b_\beta^{j_1}(x)(\nu(x))^\beta \\ \vdots \\ \sum_{|\beta|=k} b_\beta^{j_{n_k}}(x)(\nu(x))^\beta \end{pmatrix} \text{ is surjective,} \\ \text{where } \{j_i : i = 1, \dots, n_k\} = \{j : m_j = k\}. \end{array} \right. \quad (2.8)$$

Remark 2.5. In general, the normality condition (2.8) is not implied by the (L-S) condition, see e.g. [1, Remark 1.1].

We can now state the main result. In the following, the compatibility conditions read as follows. For j such that $m_j = 0$ and $x \in \partial\Omega$

$$\begin{cases} Bu_0 = G(u_0), \\ B_j(Au_0 - F(u_0)) = G'_j(u_0)(Au_0 - F(u_0)). \end{cases} \quad (2.9)$$

3 Main Result

In this section we state and prove our main theorem about convergence of solutions for the system (2.1) towards equilibria.

Theorem 3.1. *Let $u_* \equiv 0 \in X_1$ be an equilibrium of (2.1), and assume that conditions (R_1) , (R_2) , (LS) , (SP) and normality condition (2.8) are satisfied. Let A_0 defined in (2.5) denote the linearization of (2.1) at $u_* \equiv 0$, and suppose that u_* is normally stable, i.e. assume that*

- (i) *near u_* the set of equilibria \mathcal{E} is a C^2 -manifold in X_1 of dimension $k \in \mathbb{N}$,*
- (ii) *the tangent space for \mathcal{E} at u_* is given by $N(A_0)$,*
- (iii) *0 is a semi-simple eigenvalue of A_0 , i.e. $R(A_0) \oplus N(A_0) = X$,*
- (iv) *$\sigma(A_0) \setminus \{0\} \subset \mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.*

Then u_ is stable in X_1 , and there exists a $\delta > 0$ such that the unique solution $u(t)$ of (2.1) with initial value $u_0 \in X_1$, satisfying $|u_0 - u_*|_1 < \delta$ and the compatibility condition (2.9), exists on \mathbb{R}_+ and converges exponentially fast in X_1 to some $u_\infty \in \mathcal{E}$ as $t \rightarrow \infty$.*

Proof. The proof to Theorem 3.1 will be carried out in steps (a)-(g) and some intermediate results will be formulated as lemmas and propositions.

(a) Note first that due to Remark 2.2, 0 is an isolated spectral point of $\sigma(A_0)$, the spectrum of A_0 . According to assumption (iv) the spectrum $\sigma(A_0)$ admits a decomposition into two disjoint nontrivial parts with

$$\sigma(A_0) = \{0\} \cup \sigma_s, \quad \sigma_s \subset \mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}.$$

In the following, we let P^l , $l \in \{c, s\}$, denote the spectral projections for the operator A_0 with respect to the spectral sets $\sigma_c = \{0\}$ and σ_s , i.e.

$$P^c = \frac{1}{2\pi i} \int_{\gamma} R(\lambda, A_0) d\lambda \quad \text{and} \quad P^s = I - P^c \quad (3.1)$$

(see [18, Def. A.1.1]). We set $X_j^l := P^l X_j$ and $X^l := P^l X$ for $l \in \{c, s\}$ and $j \in \{0, 1\}$. The space X_j^l and X^l are equipped with the norms $|\cdot|_j$ and $|\cdot|$ respectively for $j \in \{0, 1\}$.

Lemma 3.2. $P^c|_{C^\alpha(\overline{\Omega})} \in \mathcal{L}(C^\alpha(\overline{\Omega}), C^{2m+\alpha}(\overline{\Omega}))$

Proof. At first we show $R(\lambda, A_0)|_{C^\alpha(\overline{\Omega})} : C^\alpha(\overline{\Omega}) \rightarrow C^{2m+\alpha}(\overline{\Omega})$ for $\lambda \in \rho(A_0)$. If we take $f \in C^\alpha(\overline{\Omega})$ and define $u := R(\lambda, A_0)f$, then $u \in D(A_0)$ and u solves

$$\begin{cases} (\lambda I - A)u = f \in C^\alpha(\overline{\Omega}) \\ Bu = 0 \end{cases}$$

By the elliptic regularity theory precisely Theorem 2.4 (iv) we get $u \in C^{2m+\alpha}(\overline{\Omega})$ and

$$\|u\|_{C^{2m+\alpha}(\overline{\Omega})} \leq C(\|f\|_{C^\alpha(\overline{\Omega})} + \|u\|_{C(\overline{\Omega})}).$$

In other words,

$$\|R(\lambda, A_0)f\|_{C^{2m+\alpha}(\overline{\Omega})} \leq C(\|f\|_{C^\alpha(\overline{\Omega})} + \|R(\lambda, A_0)f\|_{C(\overline{\Omega})}).$$

And now by (3.1) and the fact that $R(\lambda, A_0) \in \mathcal{L}(X, X)$, the claim follows. \square

Note that Lemma 3.2 especially implies $P^l|_{C^{2m+\alpha}(\overline{\Omega})} \subset C^{2m+\alpha}(\overline{\Omega})$ for $l \in \{c, s\}$. Let $A_l = P^l A_0 P^l$ be the part of A_0 in X^l for $l \in \{c, s\}$. Since 0 is a semi-simple eigenvalue of A_0 , we have $X^c = N(A_0)$ and $X^s = R(A_0)$ (see [18, Prop. A.2.2]) and so P^c and P^s are the projections onto $N(A_0)$ resp. $R(A_0)$. In particular it means that $A_c \equiv 0$ which is equivalent to say $AP^c \equiv 0$ and $BP^c \equiv 0$. Note that $N(A_0) \subset X_1$ by elliptic regularity precisely Theorem 2.4 (iv).

Since $X_0^c \hookrightarrow X^c \hookrightarrow X_1$ we get $X_0^c = X_1^c = X^c = N(A)$. As X^c has finite dimension, all norms are equivalent, we equip X^c with the norm of these equivalent norms say with $|\cdot|$. Moreover, we take as a norm on X_j and X

$$\begin{cases} |u|_j := |P^c u| + |P^s u|_j & \text{for } j = 0, 1 \\ |u| := |P^c u| + |P^s u| \end{cases} \quad (3.2)$$

(b) Next we show that the manifold \mathcal{E} can be represented as the graph of a function $\phi : B_{X^c}(0, \rho_0) \rightarrow X_1^s$ in a neighborhood of u_* . In order to see this we consider the mapping

$$g : U \subset \mathbb{R}^k \rightarrow X^c, \quad g(\zeta) := P^c \Psi(\zeta), \quad \zeta \in U.$$

It follows from our assumptions that $g'(0) = P^c \Psi'(0) : \mathbb{R}^k \rightarrow X^c$ is an isomorphism. By the inverse function theorem, g is a C^2 -diffeomorphism of a neighborhood of 0 in \mathbb{R}^k onto a neighborhood, say $B_{X^c}(0, \rho_0)$, of 0 in X^c . Let $g^{-1} : B_{X^c}(0, \rho_0) \rightarrow U$ be the inverse mapping. Then $g^{-1} : B_{X^c}(0, \rho_0) \rightarrow U$ is C^2 and $g^{-1}(0) = 0$. Next we set $\Phi(v) := \Psi(g^{-1}(v))$ for $v \in B_{X^c}(0, \rho_0)$ and we note that

$$\Phi \in C^2(B_{X^c}(0, \rho_0), X_1), \quad \Phi(0) = 0, \quad \{u_* + \Phi(v) : v \in B_{X^c}(0, \rho_0)\} = \mathcal{E} \cap W$$

where W is an appropriate neighborhood of u_* in X_1 . Clearly,

$$P^c \Phi(v) = ((P^c \circ \Psi) \circ g^{-1})(v) = (g \circ g^{-1})(v) = v, \quad v \in B_{X^c}(0, \rho_0),$$

and this yield $\Phi(v) = P^c \Phi(v) + P^s \Phi(v) = v + P^s \Phi(v)$ for $v \in B_{X^c}(0, \rho_0)$. Setting $\phi(v) := P^s \Phi(v)$ and using the fact that the image of $\Psi'(0)$ is contained in $N(A_0)$ we conclude that

$$\phi \in C^2(B_{X^c}(0, \rho_0), X_1^s), \quad \phi(0) = \phi'(0) = 0, \quad (3.3)$$

and that

$$\{u_* + v + \phi(v) : v \in B_{X^c}(0, \rho_0)\} = \mathcal{E} \cap W, \quad (3.4)$$

where W is a neighborhood of u_* in X_1 . This show that the manifold \mathcal{E} can be represented as the graph of the function ϕ in a neighborhood of u_* . Moreover the tangent space of \mathcal{E} at u_* coincides with $N(A_0) = X^c$. By applying the projection $P^l, l \in \{c, s\}$, to Eq. (2.3) and using that $v + \phi(v) = \Psi(g^{-1}(v))$ for $v \in B_{X^c}(0, \rho_0)$, and that $A_c \equiv 0$, we obtain the following equivalent system of equations for the equilibria of (2.1)

$$\begin{aligned} P^c A \phi(v) &= P^c F(v + \phi(v)), \\ P^s A \phi(v) &= P^s F(v + \phi(v)), \quad B \phi(v) = G(v + \phi(v)), \end{aligned} \quad (3.5)$$

for every $v \in B_{X^c}(0, \rho_0)$.

We choose ρ_0 so small that

$$|\phi'(v)|_{\mathcal{L}(X^c, X_1^s)} \leq 1, \quad |\phi(v)|_1 \leq |v|, \quad \text{for all } v \in B_{X^c}(0, \rho_0). \quad (3.6)$$

For $r \in (0, \rho_0)$, we set

$$\eta(r) = \sup\{\|\phi'(\varphi)\|_{\mathcal{L}(X^c, X_1^s)} : \varphi \in B_{X^c}(0, r)\}.$$

Since $\phi'(0) = 0$, $\eta(r)$ tends to 0 as $r \rightarrow 0$. Let $L' > 0$ be such that, for all $\varphi, \psi \in B_{X^c}(0, r)$ with $r \in (0, \rho_0)$

$$\|\phi'(\varphi) - \phi'(\psi)\|_{\mathcal{L}(X^c, X_1^s)} \leq L'|\varphi - \psi|.$$

(c) Introducing the new variables

$$v := P^c u, \quad w := P^s u - \phi(P^c u),$$

we then obtain the following system of evolution equations in $X^c \times X^s$

$$\left\{ \begin{array}{ll} \partial_t v = T(v, w) & \text{in } \Omega, \\ \partial_t w + P^s A P^s w = R(v, w) & \text{in } \Omega, \\ B w = S(v, w) & \text{on } \partial\Omega, \\ v(0) = v_0, \quad w(0) = w_0 & \text{in } \Omega, \end{array} \right. \quad (3.7)$$

with $v_0 = P^c u_0$ and $w_0 = P^s u_0 - \phi(P^c u_0)$, where the function T, R and S are given by

$$\begin{aligned} T(v, w) &= P^c F(v + \phi(v) + w) - P^c A\phi(v) - P^c Aw, \\ R(v, w) &= P^s F(v + \phi(v) + w) - P^s A\phi(v) - \phi'(v)T(v, w), \\ S(v, w) &= G(v + \phi(v) + w) - B\phi(v). \end{aligned}$$

Using the equilibrium equation (3.5), the expression for T, R and S can be rewritten as

$$\begin{aligned} T(v, w) &= P^c (F(v + \phi(v) + w) - F(v + \phi(v))) - P^c Aw, \\ R(v, w) &= P^s (F(v + \phi(v) + w) - F(v + \phi(v))) - \phi'(v)T(v, w), \\ S(v, w) &= G(v + \phi(v) + w) - G(v + \phi(v)). \end{aligned}$$

Clearly,

$$R(v, 0) = T(v, 0) = S(v, 0) = 0, \quad v \in B_{X^c}(0, \rho_0).$$

Note that here the infinite-dimensional part, i.e., the equation for w has a non-linear boundary condition in case $S(v, w) \neq 0$.

(d) Let $0 < a \leq \infty$ and define the following function spaces:

$$\mathbb{E}_1(a) = C^{1+\frac{\alpha}{2m}, 2m+\alpha}(I_a \times \overline{\Omega}) = C^{1+\frac{\alpha}{2m}}(I_a, X) \cap B(I_a, X_1),$$

$$\mathbb{E}_0(a) = C^{\frac{\alpha}{2m}, \alpha}(I_a \times \overline{\Omega}) = C^{\frac{\alpha}{2m}}(I_a, X) \cap B(I_a, X_0).$$

Here $B(I_a, X_j)$ is a space of all bounded functions $f : I_a \rightarrow X_j$ equipped with the supremum norm and

$$I_a := \begin{cases} [0, a] & \text{for } a > 0, \\ [0, \infty) & \text{for } a = \infty. \end{cases}$$

We also need spaces for the boundary values. For this purpose we set

$$\begin{aligned} \mathbb{F}_j(a) &:= C^{1+\frac{\alpha}{2m}-\frac{m_j}{2m}, 2m+\alpha-m_j}(I_a \times \partial\Omega) \\ &= C^{1+\frac{\alpha}{2m}-\frac{m_j}{2m}}(I_a, C(\partial\Omega)) \cap B(I_a, C^{2m+\alpha-m_j}(\partial\Omega)), \end{aligned}$$

and

$$\mathbb{F}(a) = \prod_{j=1}^{mN} \mathbb{F}_j(a).$$

By (3.2) you can easily show that $\|p_l u\|_{\mathbb{E}_i(a)} \leq \|u\|_{\mathbb{E}_i(a)}$ for $i = 0, 1$ and $l \in \{c, s\}$, which we will use several times without mention it. The proof of the following Lemma is given in [20, Theorem 2.2].

Lemma 3.3. *The following continuous embedding holds with an embedding constant independent of a , with $0 < \theta < 2m + \alpha$.*

$$\mathbb{E}_1(a) \hookrightarrow C^{\frac{\theta}{2m}}(I_a, C^{2m+\alpha-\theta}(\overline{\Omega})).$$

The basic solvability theorem for the fully inhomogeneous linear problem

$$\begin{cases} \partial_t u + Au = f(t) & \text{in } \Omega, \quad t \in (0, a), \\ Bu = g(t) & \text{on } \partial\Omega, \quad t \in (0, a), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \quad (3.8)$$

where $a \in (0, +\infty]$, in the parabolic Hölder setting reads as follows, see Theorem VI.21 in [11].

In the following we need the compatibility conditions

$$\begin{cases} Bu_0 = g(0), \\ B_j f(0) - B_j Au_0 = \partial_t g_j(t)|_{t=0}, \end{cases} \quad (3.9)$$

for all j such that $m_j = 0$.

Proposition 3.4. *Let $a < \infty$. The linear problem (3.8) admits a unique solution $u \in \mathbb{E}_1(a)$ if and only if $f \in \mathbb{E}_0(a)$, $g \in \mathbb{F}(a)$, $u_0 \in X_1$, and the compatibility condition (3.9) holds. There is a constant $\tilde{C} = \tilde{C}(a) > 0$ such that the estimate*

$$\|u\|_{\mathbb{E}_1(a)} \leq \tilde{C}(|u_0|_1 + \|f\|_{\mathbb{E}_0(a)} + \|g\|_{\mathbb{F}(a)})$$

holds for the solution u of (3.8).

We shall also need a variant of Proposition 3.4 for the problem

$$\begin{cases} \partial_t w + P^s A P^s w = f(t) & \text{in } \Omega, \quad t > 0, \\ Bw = g(t) & \text{on } \partial\Omega, \quad t > 0, \\ w(0) = w_0 & \text{in } \Omega, \end{cases} \quad (3.10)$$

globally in time for $t \in (0, \infty]$, where we assume $w_0 \in X_1^s$, $e^{\sigma t} f \in C^{\frac{\alpha}{2m}}(\mathbb{R}_+; X) \cap B(\mathbb{R}_+; X_0^s)$, $e^{\sigma t} g \in \mathbb{F}(\infty)$ and $0 < \sigma < \inf \{\operatorname{Re} \lambda : \lambda \in \sigma_s\}$. For this purpose we proceed as follows. Suppose first that u solves (3.8) with $u_0 = w_0$. Since $AP^c u = BP^c u = 0$, we conclude that $w = P^s u$ solves problem (3.10). Let u_1 denote the solution of (3.8) with A replaced by $A + 1$. The spectrum of $A_0 + 1$ is contained in \mathbb{C}_+ , hence we may apply Corollary 5.6 below to obtain a uniform estimate for $e^{\sigma t} u_1$ in $\mathbb{E}_1(\infty)$. There are two ways to proceed further.

1st: Then $u_2 = u - u_1$ solves the problem

$$\partial_t u_2 + Au_2 = u_1, \quad Bu_2 = 0, \quad u_2(0) = 0,$$

As $\sigma(A_s) \subset \mathbb{C}_+$, A_s has the maximal parabolic Hölder regularity on the half line, see Corollary 5.6 below, hence we obtain also a uniform estimate for $P^s u_2$ in $\mathbb{E}_1(\infty)$. These arguments yield the following result, where $\omega = \inf \{\operatorname{Re} \lambda : \lambda \in \sigma_s\}$.

2nd: Setting $u_2 = u - u_1$ we get that $z = P^s u_2$ solves the problem

$$\partial_t z + P^s A P^s z = P^s u_1, \quad Bz = 0, \quad z(0) = 0. \quad (3.11)$$

Let u_3 denote the solution of

$$\partial_t z + Az = P^s u_1, \quad Bz = 0, \quad z(0) = 0. \quad (3.12)$$

By applying Theorem 5.5 below to (3.12) with $f = P^s u_1$, $u_0 = 0$, $g = 0$ we obtain a uniform estimate for $e^{\sigma t} u_3$ in $\mathbb{E}_1(\infty)$ (it is easy to see that (5.30) holds). Now using the fact that $P^s u_3$ solves (3.11) we also obtain a uniform estimate for $e^{\sigma t} P^s u_3 = e^{\sigma t} P^s u_2$ in $\mathbb{E}_1(\infty)$. These arguments yield the following result, where $\omega = \inf \{\operatorname{Re} \lambda : \lambda \in \sigma_s\}$.

Proposition 3.5. *Let $0 < a \leq \infty$ and $0 < \sigma < \omega$. The linear problem (3.10) admits a unique solution w such that $e^{\sigma t} w \in \mathbb{E}_1(a)$ if and only if $e^{\sigma t} f \in C^{\frac{\alpha}{2m}}(I_a; X) \cap B(I_a; X_0^s)$, $e^{\sigma t} g \in \mathbb{F}(a)$, $w_0 \in X_1^s$, and the compatibility condition (3.9) holds. There is a constant C_0 , independent of a , such that the estimate*

$$\|e^{\sigma t} w\|_{\mathbb{E}_1(a)} \leq C_0(|w_0|_1 + \|e^{\sigma t} f\|_{\mathbb{E}_0(a)} + \|e^{\sigma t} g\|_{\mathbb{F}(a)})$$

holds for the solution w of (3.10), for all functions $e^{\sigma t} f \in C^{\frac{\alpha}{4}}(\mathbb{R}_+; X) \cap B(\mathbb{R}_+; X_0^s)$, $e^{\sigma t} g \in \mathbb{F}(a)$, and for all initial values $w_0 \in X_1^s$.

Proof. Proof of the "if" part is given above and to show the "only if" part, use the system of equations (3.10). \square

(e) Next we consider the nonlinearities T , R and S . Here we will derive estimates which are needed for applying Proposition 3.5.

Let $0 < r \leq R$, and set

$$\begin{aligned} K(r) &= \sup\{\|F'(\varphi)\|_{\mathcal{L}(C^{2m+\alpha}(\overline{\Omega}), C^\alpha(\overline{\Omega}))} : \varphi \in B(0, r) \subset C^{2m+\alpha}(\overline{\Omega})\}, \\ H_j(r) &= \sup\{\|G'_j(\varphi)\|_{\mathcal{L}(C^{2m+\alpha}(\overline{\Omega}), C^{2m+\alpha-m_j}(\partial\Omega))} : \varphi \in B(0, r) \subset C^{2m+\alpha}(\overline{\Omega})\}, \end{aligned}$$

for $j = 1, \dots, mN$. Since $F'(0) = 0$ and $G'_j(0) = 0$, $K(r)$ and $H_j(r)$ tend to 0 as $r \rightarrow 0$. Let $L > 0$ be such that, for all $\varphi, \psi \in B(0, r) \subset C^{2m}(\overline{\Omega})$ with small r ,

$$\begin{aligned} \|F'(\varphi) - F'(\psi)\|_{\mathcal{L}(C^{2m}(\overline{\Omega}), C(\overline{\Omega}))} &\leq L\|\varphi - \psi\|_{C^{2m}(\overline{\Omega})}, \\ \|G'_j(\varphi) - G'_j(\psi)\|_{\mathcal{L}(C^{m_j}(\overline{\Omega}), C(\partial\Omega))} &\leq L\|\varphi - \psi\|_{C^{m_j}(\overline{\Omega})}, \\ \|G''_j(\varphi) - G''_j(\psi)\|_{\mathcal{L}(C^{m_j}(\overline{\Omega}), \mathcal{L}(C^{m_j}(\overline{\Omega}), C(\partial\Omega)))} &\leq L\|\varphi - \psi\|_{C^{m_j}(\overline{\Omega})}. \end{aligned}$$

In the following, we will always assume that $r \leq \min\{R, \rho_0\}$.

Lemma 3.6. *There exist a constant C_1 such that*

$$|T(v, w)| \leq C_1|w|_1$$

for any $u \in \overline{B_{X_1}(0, r)}$.

Proof. From (3.6) we see

$$|v + \phi(v) + w|_1 = |u|_1 \leq r, \quad |v + \phi(v)|_1 \leq |v|_1 + |\phi(v)|_1 \leq 2r,$$

and now taking $z_1 = v + \phi(v) + w$ and $z_2 = v + \phi(v)$ in the definition of $T(v, w)$ we get

$$\begin{aligned} |T(v, w)| &= |P^c(F(z_1) - F(z_2))| + |P^cAw| \\ &\leq |F(z_1) - F(z_2)|_0 + \|P^cA\|_{\mathcal{L}(X_1, X^c)}|w|_1 \\ &\leq (K(2r) + C_2)|w|_1, \end{aligned}$$

where $C_2 := \|P^cA\|_{\mathcal{L}(X_1, X^c)}$. \square

Observe that the constant C_2 is no longer small since P^l , $l \in \{c, s\}$, and A do not commute.

Proposition 3.7. *If $z_1, z_2 \in \overline{B_{\mathbb{E}_1(a)}(0, r)}$, $\sigma \geq 0$ then*

$$\begin{aligned} \|e^{\sigma t}(F(z_1) - F(z_2))\|_{\mathbb{E}_0(a)} &\leq D(r)\|e^{\sigma t}(z_1 - z_2)\|_{\mathbb{E}_1(a)}, \\ \|e^{\sigma t}(G(z_1) - G(z_2))\|_{\mathbb{F}(a)} &\leq D(r)\|e^{\sigma t}(z_1 - z_2)\|_{\mathbb{E}_1(a)}, \end{aligned}$$

where $D(r) \rightarrow 0$ as $r \rightarrow 0$.

The proof is given in the appendix.

Lemma 3.8. *If $u \in \overline{B_{\mathbb{E}_1(a)}(0, r)}$, then $v + \phi(v) \in \overline{B_{\mathbb{E}_1(a)}(0, 4r + L'r^2)}$.*

Proof. For $0 \leq t \leq a$, again by (3.6), we have

$$|v(t) + \phi(v(t))|_1 \leq 2|v(t)|_1 \leq 2|u(t)|_1 \leq 2r$$

while for $0 \leq s \leq t \leq a$,

$$\begin{aligned} &|v'(t) + \phi'(v(t))v'(t) - v'(s) - \phi'(v(s))v'(s)| \\ &\leq |v'(t) - v'(s)| + |\phi'(v(t))(v'(t) - v'(s))| + |\phi'(v(t)) - \phi'(v(s))|_{\mathcal{L}(X^c, X_1)}|v'(s)| \\ &\leq 2(t-s)^{\frac{\alpha}{2m}}\|v\|_{\mathbb{E}_1(a)} + L'|v(t) - v(s)||v'(s)| \\ &\leq 2(t-s)^{\frac{\alpha}{2m}}\|v\|_{\mathbb{E}_1(a)} + L'(t-s)^{\frac{\alpha}{2m}}\|v\|_{\mathbb{E}_1(a)}^2 \\ &\leq (t-s)^{\frac{\alpha}{2m}}(2r + L'r^2) \end{aligned}$$

Note that we have used (3.6) and Lemma 3.3 to obtain the second inequality. This completes the proof. \square

Proposition 3.9. *If $u \in \overline{B_{\mathbb{E}_1(a)}(0, r)}$, $\sigma \geq 0$ then*

$$\begin{aligned} (i) \quad &\|e^{\sigma t}T(v, w)\|_{\mathbb{E}_0(a)} \leq C_3\|e^{\sigma t}w\|_{\mathbb{E}_1(a)}, \\ (ii) \quad &\|e^{\sigma t}R(v, w)\|_{\mathbb{E}_0(a)} \leq C(r)\|e^{\sigma t}w\|_{\mathbb{E}_1(a)}, \\ (iii) \quad &\|e^{\sigma t}S(v, w)\|_{\mathbb{F}(a)} \leq C(r)\|e^{\sigma t}w\|_{\mathbb{E}_1(a)}, \end{aligned}$$

where $C(r) \rightarrow 0$ as r goes to zero.

Proof. Let us prove (i). Setting $z_1 := u = v + \phi(v) + w$ and $z_2 := v + \phi(v)$ by Lemma 3.8 we have

$$\|z_1\|_{\mathbb{E}_1(a)}, \|z_2\|_{\mathbb{E}_1(a)} \leq 4r + L'r^2.$$

Hence we can now apply Proposition 3.7 to conclude

$$\begin{aligned} \|e^{\sigma t}T(v, w)\|_{\mathbb{E}_0(a)} &\leq \|P^c(e^{\sigma t}(F(z_1) - F(z_2)))\|_{\mathbb{E}_0(a)} + \|e^{\sigma t}P^cAw\|_{\mathbb{E}_0(a)} \\ &\leq \|e^{\sigma t}(F(z_1) - F(z_2))\|_{\mathbb{E}_0(a)} + \|e^{\sigma t}P^cAw\|_{\mathbb{E}_0(a)} \\ &\leq D(4r + L'r^2)\|e^{\sigma t}w\|_{\mathbb{E}_1(a)} + \|e^{\sigma t}P^cAw\|_{\mathbb{E}_0(a)} \end{aligned}$$

Now let us consider $\|e^{\sigma t}P^cAw\|_{\mathbb{E}_0(a)}$.

For $0 \leq t \leq a$,

$$|e^{\sigma t}P^cAw(t)| \leq C_2|e^{\sigma t}w(t)|_1 \leq C_2\|e^{\sigma t}w\|_{\mathbb{E}_1(0,a)}$$

while for $0 \leq s \leq t \leq a$,

$$\begin{aligned} |e^{\sigma t}P^cAw(t) - e^{\sigma s}P^cAw(s)| &\leq |A(e^{\sigma t}w(t) - e^{\sigma s}w(s))| \leq \|e^{\sigma t}w(t) - e^{\sigma s}w(s)\|_{D(A)} \\ &\leq \|e^{\sigma t}w(t) - e^{\sigma s}w(s)\|_{C^{2m}(\overline{\Omega})} \\ &\leq C'(t - s)^{\frac{\alpha}{2m}}\|e^{\sigma t}w\|_{\mathbb{E}_1(a)}, \end{aligned}$$

where we have used Lemma 3.3 to obtain the last inequality and C' is the corresponding embedding constant. Setting $C_3 := D(4r + L'r^2) + C' + C_2$ we complete the proof of (i).

We prove (ii). Similarly as in (i) we get for the first term in $e^{\sigma t}R(v, w)$

$$\|P^s(e^{\sigma t}(F(z_1) - F(z_2)))\|_{\mathbb{E}_0(a)} \leq D(4r + L'r^2)\|e^{\sigma t}w\|_{\mathbb{E}_1(a)}.$$

Let us estimate the second term in $R(v, w)$ namely, $e^{\sigma t}\phi'(v)T(v, w)$.

For $0 \leq t \leq a$, by (3.6) and Lemma 3.6 we have

$$\begin{aligned} |e^{\sigma t}\phi'(v(t))T(v(t), w(t))|_0 &\leq |e^{\sigma t}\phi'(v(t))T(v(t), w(t))|_1 \\ &\leq \|\phi'(v(t))\|_{\mathcal{L}(X^c, X_1^*)}|e^{\sigma t}T(v(t), w(t))| \\ &\leq \eta(r)|e^{\sigma t}T(v(t), w(t))| \leq C_1\eta(r)|e^{\sigma t}w(t)|_1 \\ &\leq C_1\eta(r)\|e^{\sigma t}w\|_{\mathbb{E}_1(a)}, \end{aligned}$$

while for $0 \leq s \leq t \leq a$,

$$\begin{aligned} &|e^{\sigma t}\phi'(v(t))T(v(t), w(t)) - e^{\sigma s}\phi'(v(s))T(v(s), w(s))| \\ &\leq \|\phi'(v(t))\|_{\mathcal{L}(X^c, X_1)}|e^{\sigma t}T(v(t), w(t)) - e^{\sigma s}T(v(s), w(s))| \\ &\quad + \|\phi'(v(t)) - \phi'(v(s))\|_{\mathcal{L}(X^c, X_1)}|e^{\sigma s}T(v(s), w(s))| \\ &\leq (t - s)^{\frac{\alpha}{2m}}\eta(r)\|e^{\sigma t}T(v, w)\|_{\mathbb{E}_0(a)} + C_1\|\phi'(v(t)) - \phi'(v(s))\|_{\mathcal{L}(X^c, X_1)}|e^{\sigma t}w(t)|_1 \\ &\leq (t - s)^{\frac{\alpha}{2m}}\eta(r)C_3\|e^{\sigma t}w\|_{\mathbb{E}_1(a)} + C_1L'|v(t) - v(s)|\|e^{\sigma t}w\|_{\mathbb{E}_1(a)} \\ &\leq (t - s)^{\frac{\alpha}{2m}}\eta(r)C_3\|e^{\sigma t}w\|_{\mathbb{E}_1(a)} + (t - s)^{\frac{\alpha}{2m}}C_1L'\|v\|_{E_1(a)}\|e^{\sigma t}w\|_{\mathbb{E}_1(a)} \\ &\leq (t - s)^{\frac{\alpha}{2m}}(\eta(r)C_3 + C_1L'r)\|e^{\sigma t}w\|_{\mathbb{E}_1(a)} \end{aligned}$$

Finally by defining $C(r) := \eta(r)C_3 + C_1L'r + C_1\eta(r) + D(4r + L'r^2)$ we complete the proof of (ii). Using Proposition 3.7 and Lemma 3.8, we easily get (iii). \square

(f) Now let us consider the local existence theorem for problem (2.1). Actually, since we are interested in the initial data close to the stationary solution, we could even prove existence for large time as stated in the following theorem.

Proposition 3.10. *For every $T > 0$, there are $r, \rho > 0$ such that (2.1) has a solution $u \in \mathbb{E}_1(T)$ provided $|u_0 - u_*|_1 \leq \rho$. Moreover, u is the unique solution in $B_{\mathbb{E}_1(T)}(0, r)$.*

Proof. The proof is almost exactly the same as the one in Theorem 4.1 in [19]. However we give the details.

Let $0 < r \leq R$ and define a nonlinear map

$$\Gamma : \left\{ w \in \overline{B(0, r)} \subset \mathbb{E}_1(T) : w(\cdot, 0) = u_0 \right\} \longrightarrow \mathbb{E}_1(T),$$

by $\Gamma w = v$, where v is the solution of

$$\begin{cases} \partial_t v + Av = F(w), & \text{in } \overline{\Omega} \times [0, T], \\ Bv = G(w), & \text{on } \partial\Omega \times [0, T], \\ v|_{t=0} = u_0, & \text{in } \overline{\Omega}. \end{cases}$$

Proposition 3.4, gives the estimate

$$\|v\|_{\mathbb{E}_1(T)} \leq \tilde{C}(|u_0|_1 + \|F(w)\|_{\mathbb{E}_0(T)} + \|G(w)\|_{\mathbb{F}(T)}),$$

with $\tilde{C} = \tilde{C}(T)$ in which we could assume without loss of generality that $\tilde{C} > 1$. Hence by Proposition 3.7 we have

$$\|\Gamma(w)\|_{\mathbb{E}_1(T)} \leq \tilde{C}(|u_0|_1 + 2D(r)\|w\|_{\mathbb{E}_1(T)}).$$

Therefore, if r is so small that

$$2\tilde{C}D(r) \leq \frac{1}{2}, \quad (3.13)$$

and u_0 is so small that

$$|u_0|_1 \leq \frac{r}{2\tilde{C}}, \quad (3.14)$$

Γ maps the ball $\overline{B(0, r)}$ into itself. Let us check that Γ is a $\frac{1}{2}$ -contraction. Let $w_1, w_2 \in B(0, r)$. Then we get

$$\|\Gamma w_1 - \Gamma w_2\|_{\mathbb{E}_1(T)} \leq \tilde{C}(\|F(w_1) - F(w_2)\|_{\mathbb{E}_0(T)} + \|G(w_1) - G(w_2)\|_{\mathbb{F}(T)}),$$

and again using Proposition 3.7 we have

$$\begin{aligned} \|\Gamma w_1 - \Gamma w_2\|_{\mathbb{E}_1(T)} &\leq 2\tilde{C}D(r)\|w_1 - w_2\|_{\mathbb{E}_1(T)} \\ &\leq \frac{1}{2}\|w_1 - w_2\|_{\mathbb{E}_1(T)}. \end{aligned}$$

The statement follows by the contraction mapping principle. \square

By the inequalities (3.13) and (3.14) you can easily see that for a given time T we could take r as small as we want provided $\rho < r$ is small enough. Now the strategy is as follows:

We fix a time T and choose $r \leq \min\{R, \rho_0\}$ small enough such that

$$2C_0C(r) \leq \frac{1}{2} \quad (3.15)$$

and that (3.13) holds. For such an r we have a corresponding $\rho < r$ (by Proposition 3.10). In conclusion by Proposition 3.10, problem (2.1) admits for $u_0 \in B_{X_1}(0, \rho)$ a unique solution $u \in B_{\mathbb{E}_1(T)}(0, r)$. This solution can be extended to a maximal interval of existence $[0, t_*)$. If t_* is finite, then either $u(t)$ leaves the ball $B_{X_1}(0, \rho)$ at time t_* , or the limit $\lim_{t \rightarrow t_*} u(t)$ does not exist in X_1 . We show that this cannot happen for initial values $u_0 \in B_{X_1}(0, \delta)$, with some $\delta \leq \rho$ to be chosen later.

Arguing as above, there exists some $\delta' < \frac{\rho}{2}$ such that the problem (2.1) admits for $u_0 \in B_{X_1}(0, \delta')$ a unique solution

$$u \in \overline{B_{\mathbb{E}_1(T)}(0, \frac{\rho}{2})}. \quad (3.16)$$

Suppose that $u_0 \in B_{X_1}(0, \delta)$ is given, where $\delta \leq \delta' < \rho$ will be determined later. Let t_* denote the existence time for the solution $u(t)$ of (2.1) with initial value u_0 . Let then t_1 be the exist time for the ball $B_{X_1}(0, \rho)$, i.e.

$$t_1 := \sup\{t \in (0, t_*) : |u(\tau)|_1 \leq \rho, \quad \tau \in [0, t]\}.$$

and suppose $t_1 < t_*$. Note that $t_1 \geq T$ by (3.16).

Lemma 3.11. *Under the conditions above we have $\|u\|_{\mathbb{E}_1(t_1)} \leq r$.*

Proof. Since $|u_0|_1 < \delta' < \rho$, we have $u \in B_{\mathbb{E}_1(T)}(0, r)$ by Proposition 3.10. By the definition of t_1 and the fact that $T \leq t_1$ we get $|u(T)|_1 \leq \rho$. Therefore we can now start with the initial data $u(T)$ and by finitely repeating the same process we complete the proof (since T is constant, we will get $u \in B_{\mathbb{E}_1(kT)}(0, r)$ for some k such that $kT > t_1$ and therefore the estimate follows immediately). \square

Due to (3.7), Proposition 3.5, Lemma 3.11 and Proposition 3.9 we obtain

$$\begin{aligned} \|e^{\sigma t} w\|_{\mathbb{E}_1(t_1)} &\leq C_0(|w_0|_1 + \|e^{\sigma t} R(v, w)\|_{\mathbb{E}_0(a)} + \|e^{\sigma t} S(v, w)\|_{\mathbb{F}(a)}) \\ &\leq C_0|w_0|_1 + 2C_0C(r)\|e^{\sigma t} w\|_{\mathbb{E}_1(t_1)}. \end{aligned}$$

This yields with (3.15)

$$\|e^{\sigma t} w\|_{\mathbb{E}_1(t_1)} \leq 2C_0|w_0|_1, \quad \sigma \in [0, \omega). \quad (3.17)$$

We further have for $t \in [0, t_1]$

$$|e^{\sigma t} w(t)|_1 \leq \|e^{\sigma t} w\|_{\mathbb{E}_1(t_1)} \leq 2C_0|w_0|_1,$$

which yields,

$$|w(t)|_1 \leq 2C_0 e^{-\sigma t} |w_0|_1, \quad t \in [0, t_1], \quad \sigma \in [0, \omega). \quad (3.18)$$

We deduce from the equation for v in (3.7), Lemma 3.6 that

$$\begin{aligned} |v(t)| &\leq |v_0| + \int_0^t |T(v(s), w(s))| \, ds \\ &\leq |v_0| + C_1 \int_0^t |w(s)|_1 \, ds \\ &\leq |v_0| + C_1 \int_0^\infty e^{-\sigma s} \, ds \|e^{\sigma t} w\|_{\mathbb{E}_1(t_1)} \\ &\leq |v_0| + \frac{C_1}{\sigma} \|e^{\sigma t} w\|_{\mathbb{E}_1(t_1)} \\ &\leq |v_0| + C_4 |w_0|_1, \quad t \in [0, t_1], \end{aligned}$$

where $C_4 = 2C_0 \frac{C_1}{\sigma}$. The previous estimates and (3.6) imply that for some constant $C_5 \geq 1$,

$$|u(t)|_1 \leq C_5 |u_0|_1, \quad t \in [0, t_1].$$

In particular this inequality holds for $t = t_1$. Hence choosing $\delta \leq \frac{\delta'}{2C_5}$, we have $|u(t_1)|_1 \leq \delta'/2$, a contradiction to the definition of t_1 since by (3.16) we could start with t_1 and continue further and still being in the ball $B_{X_1}(0, \rho)$ hence $t_1 = t_*$. By Lemma 3.11 we get uniform bounds $\|u\|_{\mathbb{E}_1(a)} \leq r$, for all $a < t_*$. In view of Proposition 3.10, it follows that $t_* = \infty$.

(g) Repeating the above estimates on the interval $[0, \infty)$ we obtain

$$|v(t)| \leq |v_0| + C_4 |w_0|_1, \quad |w(t)|_1 \leq 2C_0 e^{-\sigma t} |w_0|_1, \quad t \in [0, \infty),$$

for $u_0 \in B_{X_1}(0, \delta)$. Moreover,

$$\lim_{t \rightarrow \infty} v(t) = \lim_{t \rightarrow \infty} \left(v_0 + \int_0^\infty T(v(s), w(s)) \, ds \right) =: v_\infty$$

exists since the integral is absolutely convergent. This yields existence of

$$u_\infty := \lim_{t \rightarrow \infty} u(t) = \lim_{t \rightarrow \infty} v(t) + \phi(v(t)) + w(t) = v_\infty + \phi(v_\infty).$$

Clearly, u_∞ is an equilibrium for equation (2.1) by (3.4). It follows from Lemma 3.6 and the estimate (3.17) that

$$\begin{aligned} |v(t) - v_\infty| &= \left| \int_t^\infty T(v(s), w(s)) \, ds \right| \\ &\leq C_1 \int_t^\infty |w(s)|_1 \, ds \\ &\leq C_1 \int_t^\infty e^{-\sigma s} \, ds \|e^{\sigma t} w\|_{\mathbb{E}_1(\infty)} \\ &\leq C_4 e^{-\sigma t} |w_0|_1 \quad t \geq 0. \end{aligned}$$

This shows that $v(t)$ converges to v_∞ at an exponential rate. Due to (3.6) and (3.18) we get

$$\begin{aligned} |u(t) - u_\infty|_1 &= |v(t) + \phi(v(t)) + w(t) - u_\infty|_1 \\ &\leq |v(t) - v_\infty| + |\phi(v(t)) - \phi(v_\infty)|_1 + |w(t)|_1 \\ &\leq (2C_4 + 2C_0)e^{-\sigma t}|w_0|_1 \\ &\leq Ce^{-\sigma t}|P^s u_0 - \phi(P^c u_0)|_1, \end{aligned}$$

thereby completing the proof of second part of Theorem 3.1. Concerning stability, note that by Lemma 3.11 it is clear that by choosing $0 < \delta \leq \rho$ small enough, the solution starting in $B_{X_1}(u_*, \delta)$ exists on \mathbb{R}_+ and stays within $B_{X_1}(u_*, r)$. \square

Remark 3.12. *The study of the stability of other possible stationary solutions, e.g. \bar{u} can be reduced to the case of the zero stationary solution by defining the new unknown*

$$U(t) = u(t) - \bar{u}.$$

4 Stability of lens-shaped networks under surface diffusion flow

4.1 The geometric setting

The situation of the generalized principle of linearized stability in parabolic Hölder spaces may occur when studying the stability of lens-shaped networks under the surface diffusion flow. Indeed, as we will see, the set of equilibria forms a manifold and the resulting PDE has nonlocal terms.

The surface diffusion flow is a geometric evolution equation for an evolving hypersurface $\Gamma = \{\Gamma(t)\}_{t \geq 0}$ in which

$$V = -\Delta_{\Gamma(t)} k, \tag{4.1}$$

where V is the normal velocity of the surfaces, k is the the sum of the principle curvatures of the surface, and $\Delta_{\Gamma(t)}$ is the Laplace-Beltrami operator of the hypersurface $\Gamma(t)$. Our sign convention is that k is negative for spheres for which we choose the outer unit normal.

Surfaces with constant mean curvature are stationary solutions of (4.1). A natural question to ask is whether these solutions are stable under (4.1). This question has been answered positive by Elliott and Garcke [12] for circles in the plane and by Escher, Mayer and Simonett [13] for spheres in higher dimensions. In general, the surfaces will meet an outer boundary or they might intersect at triple or multiple lines.

A lens-shaped network consists of two smooth curves and two half-lines arranged as in Figure 1, that is to say, we assume that the network has reflection symmetry across the x^1 -axis and that the two curves meet the two half-lines with a constant angle $\pi - \theta$, where $0 < \theta < \pi$. Note that $\theta = \frac{\pi}{3}$ corresponds to symmetric angles at the triple junction.

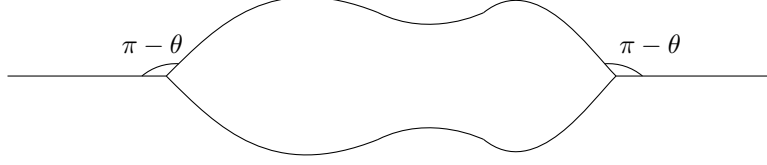


Figure 1: Lens-shaped network

More precisely, a lens-shaped network is determined by a curve Γ with the following property:

$$\begin{cases} \partial\Gamma \subset \{(x, y) \in \mathbb{R}^2 : y = 0\}, \\ \angle(N, e_2)|_{\partial\Gamma} = \theta, \end{cases}$$

where N is the unit normal to Γ pointing outwards of the bubble, see e.g. Figure 2.

Then the complete lens-shaped network is given by four curves: $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$, where Γ_1 is the curve Γ described above, Γ_2 is the reflection of Γ_1 across the x^1 -axis and Γ_3, Γ_4 are the half-lines contained in the x^1 -axis meeting Γ_1 and Γ_2 at triple junctions.

We study the following problem introduced by Garcke and Novick-Cohen [16]: Find evolving lens-shaped networks $\Gamma_1(t), \dots, \Gamma_4(t)$ as described above with the following properties:

$$\begin{cases} V_i = -\Delta_{\Gamma_i} k_i & \text{on } \Gamma_i(t), \quad t > 0, \quad (i = 1, 2, 3, 4), \\ \nabla_{\Gamma_1} k_1 \cdot n_{\partial\Gamma_1} = \nabla_{\Gamma_2} k_2 \cdot n_{\partial\Gamma_2} \\ \quad = \nabla_{\Gamma_3} k_i \cdot n_{\partial\Gamma_i} & \text{on } \partial\Gamma_i(t), \quad t > 0, \quad (i = 3, 4), \\ \Gamma_i(t)|_{t=0} = \Gamma_i^0 & (i = 1, 2, 3, 4), \end{cases} \quad (4.2)$$

where Γ_i^0 ($i = 1, 2, 3, 4$) form a given initial lens-shaped network fulfilling the balance of flux condition, i.e. the second condition in (4.2). Here V_i and k_i are the normal velocity and mean curvature of $\Gamma_i(t)$, respectively and $n_{\partial\Gamma_i}$ is the outer unit conormal of Γ_i at boundary points and ∇_{Γ_i} denotes the surface gradient of the surface $\Gamma_i(t)$.

We choose the unit normal $N_2(\cdot, t)$ of $\Gamma_2(t)$ to be pointed inwards of the bubble. With this choice of normals we observe that $k_2 = -k_1$ at the boundary points and so we get

$$k_1 + k_2 + k_i = 0 \quad \text{on } \partial\Gamma_i(t) \quad \text{for } i = 3, 4,$$

which must hold for more general triple junctions (non-symmetric, non-flat) with 120 degree angles. We refer to Garcke, Novick-Cohen [16] for the precise setting of the general problem.

Let us note that solutions to (4.2) preserve the enclosed area. Indeed, by Lemma 4.22 in [8], we have

$$\begin{aligned}
\frac{d}{dt} \int_{\Omega(t)} 1 \, dx &= - \int_{\Gamma_2(t)} V_2 \, ds + \int_{\Gamma_1(t)} V_1 \, ds \\
&= \int_{\Gamma_2(t)} \Delta_{\Gamma_2} k_2 \, ds - \int_{\Gamma_1(t)} \Delta_{\Gamma_1} k_1 \, ds \\
&= \int_{\partial\Gamma_2(t)} \nabla_{\Gamma_2} k_2 \cdot n_{\partial\Gamma_2} \, ds - \int_{\partial\Gamma_1(t)} \nabla_{\Gamma_1} k_1 \cdot n_{\partial\Gamma_1} \, ds \\
&= 0,
\end{aligned}$$

where $\Omega(t)$ is defined as the region bounded by $\Gamma_1(t)$ and $\Gamma_2(t)$.

Using the fact that the curvature of $\Gamma_3(t)$ and $\Gamma_4(t)$ are zero, it is easy to verify that the family of lens-shaped networks $\Gamma_1(t), \dots, \Gamma_4(t)$ evolves according to (4.2) if $\Gamma(t) := \Gamma_1(t)$ satisfies

$$\begin{cases} V = -\Delta_{\Gamma} k & \text{on } \Gamma(t), \quad t > 0, \\ \partial\Gamma(t) \subset \{(x, y) \in \mathbb{R}^2 : y = 0\} & t > 0, \\ N \cdot e_2 = \cos \theta & \text{on } \partial\Gamma(t), \quad t > 0, \\ \nabla_{\Gamma} k \cdot n_{\partial\Gamma} = 0 & \text{on } \partial\Gamma(t), \quad t > 0, \\ \Gamma(t)|_{t=0} = \Gamma_0, \end{cases} \quad (4.3)$$

where Γ_0 is a given initial curve which fulfills the contact, angle and no-flux condition as above.

Remark 4.1. *The equation $V = -\Delta_{\Gamma} k$ written in a local parameterization is a fourth order parabolic equation and above we prescribe three boundary conditions. This is due to the fact that (4.3) is a free boundary problem because the points in $\partial\Gamma$ can move in the set $\{y = 0\}$, see [6] for a related second order problem. Moreover, we would like to refer to the work of Schnürer and co-authors [25], where they consider the evolution of symmetric convex lens-shaped networks under the curve shortening flow.*

Let us look at equilibria of the problem (4.3). It is easy to verify that the curvature of the stationary solutions is constant and so the set of equilibrium states of (4.3) consists precisely of all circular arcs that intersect the x -axis with $\pi - \theta$ degree angles denoted by $SA_r(z_1, -r \cos \theta)$, where $|r|$ denotes the radius and $(z_1, -r \cos \theta)$ are the coordinates of the center, with $z_1 \in \mathbb{R}$, $r \in \mathbb{R} \setminus \{0\}$ (see Figure 2 for the justification of the coordinates of the center). Therefore the set of equilibria forms a 2-parameter family, the parameters are the radius of the circular arc and the first component of the center.

It is a goal of this section to prove the stability of such stationary solutions (see Theorem 4.3) using the generalized principle of linearized stability in parabolic Hölder spaces, i.e. Theorem 3.1.

Let us briefly outline how we proceed. At first we parameterize the curves around a stationary curve with the help of a modified distance function introduced in Depner and Garcke [9]. We also note that the linearization is

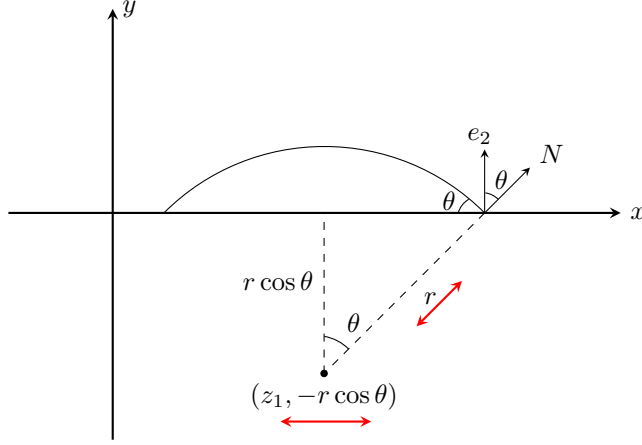


Figure 2: Circular arcs $SA_r(z_1, -r \cos \theta)$ for $r > 0$

completely done in [9] when using this modified distance function in order to formulate the governing PDE. We then formulate the evolution problem with the help of this parameterization and derive a highly nonlinear, nonlocal problem (4.8).

In Section 4.3, after deriving the linearization around the stationary solution, we show how our nonlinear, nonlocal problem fits into our abstract evolution system (2.1). We then continue by checking the assumption (R_1) , (R_2) , (LS), (SP) and the normality condition (2.8).

Finally, in order to apply Theorem 3.1, it remains to check the assumption that the stationary solution is normally stable which is done in Section 4.4.

4.2 Parameterization and PDE formulation

4.2.1 Parameterization

In this section we introduce the mathematical setting in order to reformulate our geometric evolution law, i.e. (4.3) as a partial differential equation for an unknown function. To this end, we use a parameterization with two parameters corresponding to a movement in tangential and normal direction, introduced in Depner and Garcke [9], see also [10].

Let us describe $\Gamma(t)$ with the help of a function $\rho : \Gamma_* \times [0, T) \rightarrow \mathbb{R}$ as graphs over some fixed stationary solution Γ_* . Note that the curvature k_* of Γ_* is constant and negative and the length of Γ_* is $2l_*$, where

$$-k_* l_* = \theta.$$

Let x be the arc-length parameter of Γ_* . Then we denote an arc-length parameterization of Γ_* as

$$\Gamma_* = \{\Phi_*(x) \mid x \in [-l_*, l_*]\}.$$

with unit tangent vector T_* .

For $\sigma \in \Gamma_*$, we set $\Phi_*^{-1}(\sigma) = x(\sigma) \in \mathbb{R}$. Here and hereafter, for simplicity, we use the notation

$$w(\sigma) = w(x) \quad (\sigma \in \Gamma_*),$$

i.e. we omit the parameterization. In particular, we set $\partial_\sigma w := \partial_x(w \circ \Phi_*)$.

To parameterize a curve close to Γ_* , we define the mapping through

$$\begin{aligned} \Psi : \Gamma_* \times (-\epsilon, \epsilon) \times (-\delta, \delta) &\rightarrow \mathbb{R}^2, \\ (\sigma, w, r) &\mapsto \Psi(\sigma, w, r) := \sigma + wN_*(\sigma) + r\tau_*(\sigma), \end{aligned} \quad (4.4)$$

where τ_* is a tangential vector field on Γ_* with support in a neighbourhood of $\partial\Gamma_*$, which equals the outer unit conormal $n_{\partial\Gamma_*}$ at $\partial\Gamma_*$.

For functions

$$\rho : \Gamma_* \times [0, T] \rightarrow (-\epsilon, \epsilon), \quad \mu : \partial\Gamma_* = \{a_*, b_*\} \times [0, T] \rightarrow (-\delta, \delta) \quad (4.5)$$

we define the mapping $\Phi = \Phi_{\rho, \mu}$ (we often omit the subscript (ρ, μ) for shortness) through

$$\Phi : \Gamma_* \times [0, T] \rightarrow \mathbb{R}^2, \quad \Phi(\sigma, t) := \Psi(\sigma, \rho(\sigma, t), \mu(\text{pr}(\sigma), t)). \quad (4.6)$$

Herein $\text{pr} : \Gamma_* \rightarrow \partial\Gamma_* = \{a_*, b_*\}$ is defined such that $\text{pr}(x) \in \partial\Gamma_*$ is the point on $\partial\Gamma_*$ with shortest distance on Γ_* to σ . We remark here that pr is well-defined and smooth close to $\partial\Gamma_*$. Note that we need this mapping just in a (small) neighbourhood of $\partial\Gamma_*$ because it is used in the product $\mu(\text{pr}(\sigma), t)\tau_*(\sigma)$, where the second term is zero outside a (small) neighbourhood of $\partial\Gamma_*$. For small $\epsilon, \delta > 0$ and fixed t we set

$$(\Phi)_t : \Gamma_* \rightarrow \mathbb{R}^2, \quad (\Phi)_t(\sigma) := \Phi(\sigma, t),$$

and finally we define a new curve through

$$\Gamma_{\rho, \mu}(t) := \text{image}((\Phi)_t).$$

We observe that for $\rho \equiv 0$ and $\mu \equiv 0$ the resulting curve is simply $\Gamma_{\rho \equiv 0, \mu \equiv 0}(t) = \Gamma_*$ for every t .

In the definition of Ψ we allow at the boundary for a movement in normal and tangential direction, and hence there are enough degrees of freedom to formulate the condition, that the curve $\Gamma(t)$ meets the x -axis at its boundary, through

$$\Phi(\sigma, t) \cdot e_2 = 0 \quad \text{for } \sigma \in \partial\Gamma_*, t \geq 0.$$

Using the definition of Φ , the fact that $\sigma \cdot e_2 = 0$ on $\partial\Gamma_*$ and the angle condition, we easily get

$$\mu = -\frac{N_* \cdot e_2}{n_{\partial\Gamma_*} \cdot e_2} \rho = -\left(\frac{\cos \theta}{\cos(\frac{\pi}{2} + \theta)}\right) \rho = (\cot \theta) \rho \quad \text{on } \partial\Gamma_*. \quad (4.7)$$

We assume that the initial curve Γ_0 from (4.3) is also given as a graph over Γ_* , i.e.

$$\Gamma_0 = \text{image}\{\sigma \mapsto \Psi(\sigma, \rho_0(\sigma), \mu_0(\text{pr}(\sigma))) \mid \sigma \in \Gamma_*\}.$$

Herein we assume $\rho_0 \in C^{4+\alpha}(\Gamma_*)$ with $\|\rho_0\|_{C^{4+\alpha}} \leq \epsilon$ for some small $\epsilon > 0$. Since Γ_0 is assumed to satisfy the contact and angle condition, it follows that $\mu_0 = (\cot \theta)\rho_0$ at $\partial\Gamma_*$.

4.2.2 The nonlocal, nonlinear parabolic boundary value problem

Following the discussion in [10, Section 2.2] and taking into account (4.7), by writing problem (4.3) with the help of the functions ρ, μ , see (4.5), over the fixed curve Γ_* and recalling the parameterization, see (4.6), we are led to the following nonlinear, nonlocal problem (see [10, Eq. (20)] for the analogous result obtained for the mean curvature flow):

$$\begin{cases} \partial_t \rho(x, t) = \mathcal{F}\left(x, \rho(x, t), \partial_x^1 \rho(x, t), \dots, \partial_x^4 \rho(x, t) \dots \right. \\ \quad \left. \dots \rho(\pm l_*, t), \partial_x^1 \rho(\pm l_*, t), \dots, \partial_x^4 \rho(\pm l_*, t) \right) & \text{for } x \in [-l_*, l_*], \\ 0 = \mathcal{G}_1(x, \rho(x, t), \partial_x^1 \rho(x, t)) & \text{at } x = \pm l_*, \\ 0 = \mathcal{G}_2(x, \rho(x, t), \partial_x^1 \rho(x, t), \partial_x^2 \rho(x, t), \partial_x^3 \rho(x, t)) & \text{at } x = \pm l_*, \\ \rho(x, 0) = \rho_0(x) & \text{for } x \in [-l_*, l_*], \end{cases} \quad (4.8)$$

where the term $\pm l_*$ should be understood in a sense that $+l_*$ is taken in (4.8) for the values of x in the neighbourhood of l_* and $-l_*$ is taken in (4.8) for the values of x in the neighbourhood of $-l_*$.

Note that the functions $\mathcal{F}, \mathcal{G}_1, \mathcal{G}_2$ are smooth with respect to the first variable and also smooth with respect to the ρ -dependent variables in some neighbourhood of $\rho \equiv 0$. Indeed these are rational functions with smooth coefficients in the ρ -dependent variables (possibly inside of square roots which are equal to 1 at $\rho \equiv 0$) with nonzero denominator at $\rho \equiv 0$ (see the discussion in [10, p. 26]).

Remark 4.2. *Exactly at this point one needs to use the classical setting, e.g. the parabolic Hölder setting rather than the L_p -setting because of the nonlocal term $\partial_x^4 \rho(\pm l_*, t)$, see (4.8), which can not be defined in an L_p -setting.*

4.3 Linearization

For the linearization of (4.8) around $\rho \equiv 0$, that is around the stationary solution Γ_* , we refer to [9] (see also [10]). More precisely, the linearization of surface diffusion is done in [9, Lemma 3.2] and a similar argument as in [9, Lemma 3.4] gives the following linearization of the angle condition

$$\partial_{n_{\partial\Gamma_*}} \rho + k_{n_{\partial\Gamma_*}} \mu = 0 \quad \text{on } \partial\Gamma_*.$$

Altogether, using the following facts (remind that x is the arc-length parameter of Γ_* and T_* is the unit tangential vector)

$$\begin{aligned}\Delta_{\Gamma_*}\rho &= \partial_x^2\rho, \\ \partial_{n_{\partial\Gamma_*}}\rho &= \nabla_{\Gamma_*}\rho \cdot n_{\partial\Gamma_*} = \partial_x\rho (T_* \cdot n_{\partial\Gamma_*}) = \pm\partial_x\rho \quad \text{at } x = \pm l_*, \\ k_{n_{\partial\Gamma_*}} &= k_*, \\ \mu &= \cot\theta\rho,\end{aligned}$$

we get for the linearization of (4.8) around $\rho \equiv 0$ the following linear equation for ρ

$$\begin{cases} \partial_t\rho + \partial_x^2(\partial_x^2 + k_*^2)\rho &= 0 & \text{for } x \in [-l_*, l_*], \\ \pm\partial_x\rho + k_*(\cot\theta)\rho &= 0 & \text{at } x = \pm l_*, \\ \partial_x(\partial_x^2 + k_*^2)\rho &= 0 & \text{at } x = \pm l_*. \end{cases} \quad (4.9)$$

Now we are able to rewrite the nonlinear, nonlocal problem (4.8) as a perturbation of a linearized problem, that is of the form (2.1), where $\Omega = (-l_*, l_*)$, the operator A is given by

$$(Au)(x) = \partial_x^2(\partial_x^2 + k_*^2)u(x), \quad x \in [-l_*, l_*],$$

the B_j 's are given by

$$(B_1u)(x) = \pm\partial_xu(x) + k_*(\cot\theta)u(x), \quad x = \pm l_*,$$

$$(B_2u)(x) = \partial_x(\partial_x^2 + k_*^2)u(x), \quad x = \pm l_*.$$

The corresponding F is a regular function defined in a neighbourhood of 0 in $C^4(\overline{\Omega})$ with values in $C(\overline{\Omega})$. Indeed, it is Frechet-differentiable of arbitrary order in a neighbourhood of zero (using the differentiability of composition operators, see e.g. Theorem 1 and 2 of [24, Sect. 5.5.3]) and the similar argument works for the corresponding functions G_1 and G_2 . In particular, the assumption (R_1) for sufficiently small $R > 0$ is satisfied.

Clearly, the operators A, B_1, B_2 satisfy the assumption (R_2) , the operator A is uniformly strongly parabolic and the operators $B = (B_1, B_2)$ satisfy the normality condition (2.8).

Let us verify that the linearized problem satisfies the complementarity condition, i.e. (L-S). For $x = \pm l_*$ and $\lambda \in \overline{\mathbb{C}_+}$, $\lambda \neq 0$ we should consider the following ODE

$$\begin{cases} \lambda v(y) + \partial_y^4 v(y) = 0, & y > 0, \\ \partial_y v(0) = 0, & \partial_y^3 v(0) = 0, \end{cases} \quad (4.10)$$

and prove that $v = 0$ is the only solution which vanishes at infinity. This can be done by the energy method. Testing the first line in (4.10) with v and using the boundary conditions and the fact that v and therefore its derivatives vanish

at infinity (since solutions of (4.10) are the linear combinations of exponential functions) we obtain

$$\begin{aligned} 0 &= \lambda \int_0^\infty v^2 dy + \int_0^\infty v \partial_y^4 v dy \\ &= \lambda \int_0^\infty v^2 dy - \int_0^\infty \partial_y v \partial_y^3 v dy \\ &= \lambda \int_0^\infty v^2 dy + \int_0^\infty (\partial_y^2 v)^2 dy. \end{aligned}$$

Since $0 \neq \lambda \in \overline{\mathbb{C}_+}$, v has to be zero and so the claim follows.

Concerning the compatibility condition, as we have assumed that the initial curve satisfies the contact, angle, and no-flux conditions, we get at $\rho = \pm l_*$

$$\begin{cases} \mathcal{G}_1(x, \rho_0(x, t), \partial_x^1 \rho_0(x, t)) = 0, \\ \mathcal{G}_2(x, \rho_0(x, t), \partial_x^1 \rho_0(x, t), \partial_x^2 \rho_0(x, t), \partial_x^3 \rho_0(x, t)) = 0, \end{cases} \quad (4.11)$$

which is equivalent to the corresponding compatibility condition (2.9) since we don't have zero order boundary conditions.

4.4 $\rho \equiv 0$ is normally stable

Here in this section, we will show that $\rho \equiv 0$, which corresponds to Γ_* , is normally stable, i.e. it satisfies the assumption (i)-(iv) in Theorem 3.1.

To begin with, let us consider the eigenvalue problem for A_0 (see (2.5) for the precise definition of A_0) which reads as follows

$$\begin{cases} \lambda u - \partial_x^2(\partial_x^2 + k_*^2)u = 0 & \text{in } [-l_*, l_*], \\ \pm \partial_x u + k_* \cot \theta u = 0 & \text{at } x = \pm l_*, \\ \partial_x(\partial_x^2 + k_*^2)u = 0 & \text{at } x = \pm l_*, \end{cases} \quad (4.12)$$

where $u \in D(A_0)$. The inner product of the first line in (4.12) with $(\partial_x^2 + k_*^2)u$ yields, after an integration by parts,

$$-\lambda I(u, u) + \int_{-l_*}^{l_*} (\partial_x(\partial_x^2 + k_*^2)u)^2 dx = 0, \quad (4.13)$$

where

$$I(u, u) = \int_{-l_*}^{l_*} (\partial_x u)^2 dx - k_*^2 \int_{-l_*}^{l_*} u^2 dx + k_* \cot \theta (u^2(l_*) + u^2(-l_*)).$$

Interestingly, the same bilinear form appears in [14, p. 1040] (taking $h_+ = h_- = k_* \cot \theta$ in [14]).

Let us first consider the case where $\lambda \neq 0$. The positivity of $I(u, u)$ is shown in [14, Sec. 7], indeed we have

$$h = k_* \cot \theta = k_* \cot(-k_* l_*) = -\frac{k_*}{\tan(k_* l_*)},$$

which is the same equality as in [14, p. 1053]. Now (4.13) implies that all eigenvalues except zero are positive, in other word the operator A_0 satisfies the assumption (iv) in Theorem 3.1.

For $\lambda = 0$ (4.13) implies $\partial_x^2 u + k_*^2 u = c$, where c is a constant. It follows that $u = a \sin(k_* x) + b \cos(k_* x) + c$, where a, b are constants. Applying the boundary conditions we get $b = -c \cos \theta$ and therefore we obtain a 2- dimensional eigenspace for the eigenvalue $\lambda = 0$. In other words,

$$N(A_0) = \text{span} \{ \sin(k_* x), 1 - (\cos \theta) \cos(k_* x) \}.$$

Let us prove that near $\rho \equiv 0$, which corresponds to Γ_* , the set \mathcal{E} of equilibria of the nonlinear problem creates a manifold (see (2.2) for the precise definition of \mathcal{E}). Note that

$$\mathcal{E} = \left\{ \rho \text{ s.t } S \in SA_r(z_1, -r \cos \theta) \text{ is parameterized over } \Gamma_* \text{ by } \rho \right\}$$

by the fact that $V \equiv 0$ if and only if $\rho_t \equiv 0$. Without loss of generality we may assume that Γ_* is centered at the origin of \mathbb{R}^2 . Suppose that $S \in SA_r(z_1, -r \cos \theta)$ which is sufficiently close to Γ_* . Using our parameterization, we see that S can be parameterized over Γ_* by the function ρ satisfying the following identity

$$\| \Psi(\sigma, \rho, \mu) - (z_1, -r \cos \theta) \|^2 - r^2 = 0. \quad (4.14)$$

By differentiating (4.14) with respect to r and evaluating it at $(0, r_*)$, which corresponds to Γ_* , we get

$$\left(\partial_r \rho(0, r_*) N_*(\sigma) + \partial_r \mu(0, r_*) \tau_*(\text{pr}(\sigma)) - \begin{pmatrix} 0 \\ -\cos \theta \end{pmatrix} \right) \cdot \left(\sigma - \begin{pmatrix} 0 \\ -r_* \cos \theta \end{pmatrix} \right) - r_* = 0.$$

Using the fact that $\sigma - \begin{pmatrix} 0 \\ -r_* \cos \theta \end{pmatrix} = r_* N_*(\sigma)$ (see Figure 3) and that τ_* is a tangential vector field, we get

$$r_* \partial_r \rho(0, r_*) + \cos \theta (\sigma_2 + r_* \cos \theta) = r_*.$$

By writing it in spherical coordinates, i.e.

$$\sigma = (\sigma_1, \sigma_2) = \Phi_*(x) = \left(r_* \sin\left(\frac{x}{r_*}\right), r_* \cos\left(\frac{x}{r_*}\right) - r_* \cos \theta \right)$$

we obtain

$$\partial_r \rho(0, r_*) = 1 - \cos \theta \cos(k_* x).$$

Analogously, we get $\partial_{z_1} \rho(0, r_*) = -\sin(k_* x)$ and so we see that near Γ_* the set \mathcal{E} of equilibria of the nonlinear problem is a smooth manifold of dimension 2, and that the tangent space $T_{\Gamma_*} \mathcal{E}$ of \mathcal{E} at Γ_* coincide with $N(A_0)$.

In order to apply Theorem 3.1 it remains to verify that the eigenvalue 0 of A_0 is semi-simple. Since the operator A_0 has compact resolvent (see Remark (2.2)),

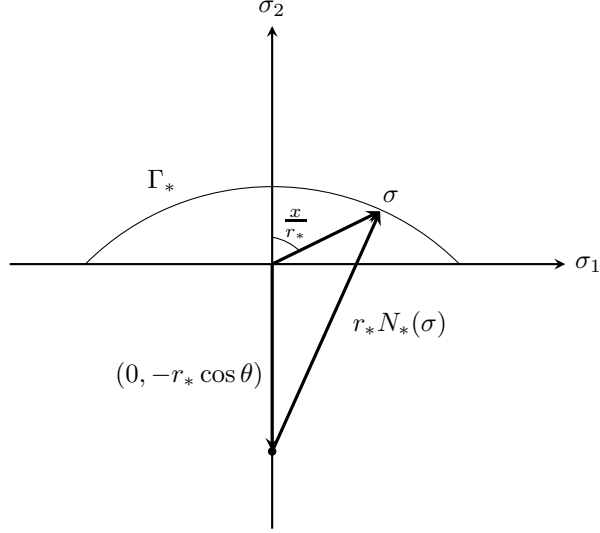


Figure 3: The stationary solution Γ_*

the semi-simplicity condition is equivalent to the condition that $N(A_0) = N(A_0^2)$ (using the spectral theory of compact operators, e.g. see [4, Sec. 9.9]). To show the latter, it can be easily seen that it is sufficient to prove the existence of a projection

$$P : X \rightarrow \mathcal{R}(P) = N(A_0)$$

such that P commutes with A_0 , that is, $PA_0u = A_0Pu (= 0)$ for all $u \in D(A_0)$.

Indeed we can construct such a projection in a following way:

$$P : X \rightarrow N(A_0) : u \mapsto Pu := \alpha_1(u)v_1 + \alpha_2(u)v_2, \quad (4.15)$$

where

$$v_1 = 1 - \cos \theta \cos(k_* x), \quad v_2 = \sin(k_* x),$$

$$\alpha_1(u) = \frac{\int_{-l_*}^{l_*} u(x) dx}{\int_{-l_*}^{l_*} v_1(x) dx}, \quad \alpha_2(u) = \frac{(u - \alpha_1(u)v_1, v_2)_{-1}}{(v_2, v_2)_{-1}}.$$

Here, the inner product is defined as

$$(\rho_1, \rho_2)_{-1} := \int_{-l_*}^{l_*} \partial_x u_{\rho_1} \partial_x u_{\rho_2},$$

where $u_{\rho_i} \in H^1(-l_*, l_*)$ for a given $\rho_i \in (H^1(-l_*, l_*))'$ with $\langle \rho_i, 1 \rangle = 0$ satisfies

$$\langle \rho_i, \xi \rangle = \int_{-l_*}^{l_*} \partial_x u_{\rho_i} \partial_x \xi$$

for all $\xi \in H^1(-l_*, l_*)$ (see [14, Sec. 4] for more detail). Here we denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $(H^1(-l_*, l_*))'$ and $(H^1(-l_*, l_*))$.

Since $\int v_1 \neq 0$, $\int v_2 = 0$ and $\int u - \alpha_1(u)v_1 dx = 0$, the coefficients $\alpha_1(u), \alpha_2(u)$ are well defined and moreover $\alpha_i(v_j) = \delta_{ij}$. Therefore P acts identity on its image $N(A_0)$ or equivalently we get $P^2 = P$.

Furthermore, for $u \in D(A_0)$ we have

$$\begin{aligned}\alpha_1(A_0 u) &= \frac{\int_{-l_*}^{l_*} A_0 u(x) dx}{\int_{-l_*}^{l_*} v_1(x) dx} = \frac{\int_{-l_*}^{l_*} \partial_x^2 (\partial_x^2 + k_*^2) u dx}{\int_{-l_*}^{l_*} v_1(x) dx} = \frac{\partial_x (\partial_x^2 + k_*^2) u|_{-l_*}^{l_*}}{\int_{-l_*}^{l_*} v_1(x) dx} = 0, \\ \alpha_2(A_0 u) &= \frac{(A_0 u, v_2)_{-1}}{(v_2, v_2)_{-1}} = \frac{(u, A_0 v_2)_{-1}}{(v_2, v_2)_{-1}} = 0,\end{aligned}$$

where we have used the facts that $v_2 \in N(A_0)$ and the operator A_0 is symmetric with respect to the inner product $(\cdot, \cdot)_{-1}$ (see [14, Lemma 5.1]). Therefore

$$PA_0 u = \alpha_1(A_0 u)v_1 + \alpha_2(A_0 u)v_2 = 0$$

and having found this projection we completed the proof of the assumption (iii) in Theorem 3.1.

So all the assumptions of Theorem 3.1 are satisfied, hence we obtain the following result:

Theorem 4.3. *Suppose Γ_* is an arbitrary circular arc with the corresponding angle θ . Assume that the initial value $\rho_0 \in X_1 = C^{4+\alpha}([-l_*, l_*])$ satisfies the compatibility condition (4.11). Then $\rho \equiv 0$ is a stable equilibrium of (4.8) in X_1 , and there exists $\delta > 0$ such that if $\|\rho_0\|_{X_1} < \delta$, then the corresponding solution of (4.8) exists globally, i.e. $\rho \in C^{1+\frac{\alpha}{2}, 4+\alpha}([0, \infty) \times [-l_*, l_*])$ and converges at an exponential rate in X_1 to some equilibrium ρ_∞ as $t \rightarrow \infty$.*

In this sense, the lens-shaped network generated by Γ_ , denoted by LS is stable under the surface diffusion flow and any lens-shaped solutions of (4.2) that starts sufficiently close to LS and satisfies a balance of flux condition at $t = 0$ exists globally and converges to some lens-shaped network generated by circular arc at an exponential rate as $t \rightarrow \infty$.*

5 Appendix

Throughout the appendix we use the notation of the previous sections.

We denote by $\sigma^-(-A_0)$ the subset of $\sigma(-A_0)$ consisting of elements with negative real parts. Since $\sigma(-A_0)$ is discrete, $\sigma^-(-A_0)$ is a spectral set (see [18, Def. A.1.1]). Let P^- be the spectral projection associated to $\sigma^-(-A_0)$. Note that $\sigma^-(-A_0) = -\sigma_s$ and $P^- = P^s$.

5.1 Asymptotic behavior in linear equations

Such a result is proven in [7, Thm. 0.1] for a single equation of second order with first order boundary condition. Here we extend this result to the systems of m

boundary conditions for a single equation of order $2m$. Precisely we consider the linear problem (3.8) with $\mathbf{N} = \mathbf{1}$, i.e.

$$\begin{cases} \partial_t u + Au = f(t) & \text{in } \Omega, \quad t \geq 0, \\ Bu = g(t) & \text{on } \partial\Omega, \quad t \geq 0, \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \quad (5.1)$$

where Ω is a bounded open set in \mathbb{R}^n with $C^{2m+\alpha}$ boundary, $0 < \alpha < 1$, $g = (g_1, \dots, g_m)$, $B = (B_1, \dots, B_m)$, $u_0 \in C^{2m+\alpha}(\overline{\Omega})$ and the operators A and B satisfy the (R_2) , (L-S), (SP), and normality condition (2.8) which in particular implies that $0 \leq m_1 < m_2 < \dots < m_m \leq 2m - 1$.

For convenience, we set

$$\mathcal{L} = -A \quad \text{and} \quad L = -A_0.$$

The realisation L of \mathcal{L} with homogeneous boundary conditions in $X = C(\overline{\Omega})$, defined similarly as (2.5), is a sectorial operator by Theorem 2.4. Moreover if f, g and u_0 are regular enough the unique solution of (5.1) is given by the extension of the Balakrishnan formula with some adaptations (see (37)-(40) of §7 in [20])

$$\begin{aligned} u(\cdot, t) = & e^{tL}(u_0 - n(\cdot, 0)) + \int_0^t e^{(t-s)L}[f(\cdot, s) + \mathcal{L}n(\cdot, s) - n'_1(\cdot, 0)]ds \\ & + n_1(\cdot, t) - \int_0^t e^{(t-s)L}(n'_1(\cdot, s) - n'_1(\cdot, 0))ds \\ & + n_2(\cdot, 0) - L \int_0^t e^{(t-s)L}[n_2(\cdot, s) - n_2(\cdot, 0)]ds \end{aligned} \quad (5.2)$$

$$\begin{aligned} = & e^{tL}u_0 + \int_0^t e^{(t-s)L}[f(\cdot, s) + \mathcal{L}n(\cdot, s)]ds \\ & - L \int_0^t e^{(t-s)L}n(\cdot, s)ds, \quad 0 \leq t \leq T. \end{aligned} \quad (5.3)$$

Here

$$\begin{aligned} n(t) = \mathcal{N}(g_1(t), \dots, g_m(t)) &= \sum_{s=1}^m \mathcal{N}_s \mathcal{M}_s(g_1(t), \dots, g_s(t)), \\ \begin{cases} n_1(t) = \begin{cases} 0 & \text{if } m_1 > 0 \\ \mathcal{N}_1 \mathcal{M}_1(g_1(t)) & \text{if } m_1 = 0, \end{cases} \\ n_2(t) = n(t) - n_1(t), \end{cases} \end{aligned} \quad (5.4)$$

where the operator \mathcal{N} is a lifting operator with an explicit construction, presented in the following theorem which is proven in [20, Thm. 6.3], such that

$$\begin{cases} \mathcal{N} \in L(\prod_{j=1}^m C^{2m+\theta'-m_j}(\partial\Omega), C^{2m+\theta'}(\overline{\Omega})), \quad \forall \theta' \in [0, \alpha], \\ B_j(\mathcal{N}(g_1, \dots, g_m))(x) = g_j(x), \quad x \in \partial\Omega, \quad j = 1, \dots, m. \end{cases} \quad (5.5)$$

In the following $\mathbb{N} = \{0, 1, 2, 3, \dots\}$.

Theorem 5.1. *Given $s = 1, \dots, m$, there exist*

$$\mathcal{M}_s \in L\left(\prod_{j=1}^s C^{\theta-m_j}(\partial\Omega), C^{\theta-m_s}(\partial\Omega)\right), \quad \forall \theta \in [m_s, 2m + \alpha],$$

and

$$\mathcal{N}_s \in L(C^r(\partial\Omega); C^{r+m_j}(\bar{\Omega})), \quad \forall r \in [0, 2m + \alpha - m_j]$$

such that, setting

$$\mathcal{N}(\psi_1, \dots, \psi_m) = \sum_{s=1}^m \mathcal{N}_s \mathcal{M}_s(\psi_1, \dots, \psi_s),$$

we have

$$\mathcal{N} \in L\left(\prod_{j=1}^m C^{2m+\theta'-m_j}(\partial\Omega), C^{2m+\theta'}(\bar{\Omega})\right), \quad \forall \theta' \in [0, \alpha],$$

and

$$B_j(\mathcal{N}(\psi_1, \dots, \psi_m))(x) = \psi_j(x), \quad x \in \partial\Omega, \quad j = 1, \dots, m.$$

Moreover, for each $u \in C(\partial\Omega)$,

$$D_x^l \mathcal{N}_s u(x) = 0, \quad x \in \partial\Omega, \quad l \in \mathbb{N}^n, \quad |l| < m_s, \quad (5.6)$$

which in particular implies that

$$(B_j \mathcal{N}_s u)(x) \equiv 0, \quad x \in \partial\Omega, \quad \text{for } j < s.$$

Theorem 5.2. *Let $0 < \omega < -\max\{\operatorname{Re}\lambda : \lambda \in \sigma^-(A_0)\}$. Let f be such that $(\sigma, t) \mapsto e^{\omega t} f(\sigma, t) \in \mathbb{E}_0(\infty)$, let g be such that $(\sigma, t) \mapsto e^{\omega t} g(\sigma, t) \in \mathbb{F}(\infty)$ and let $u_0 \in C^{2m+\alpha}(\bar{\Omega})$ satisfy the compatibility condition (3.9). Then $v(\sigma, t) = e^{\omega t} u(\sigma, t)$ is bounded in $[0, +\infty) \times \bar{\Omega}$ if and only if*

$$\begin{aligned} (I - P^-)u_0 = & - \int_0^{+\infty} e^{-sL} (I - P^-)[f(\cdot, s) + \mathcal{L}\mathcal{N}g(\cdot, s)] ds \\ & + L \int_0^{\infty} e^{-sL} (I - P^-)\mathcal{N}g(\cdot, s) ds. \end{aligned} \quad (5.7)$$

In this case, u is given by

$$\begin{aligned} u(\cdot, t) = & e^{tL} P^- u_0 + \int_0^t e^{(t-s)L} P^- [f(\cdot, s) + \mathcal{L}\mathcal{N}g(\cdot, s)] ds \\ & - L \int_0^t e^{(t-s)L} P^- \mathcal{N}g(\cdot, s) ds \\ & - \int_t^{+\infty} e^{(t-s)L} (I - P^-)[f(\cdot, s) + \mathcal{L}\mathcal{N}g(\cdot, s)] ds \\ & + L \int_t^{+\infty} e^{(t-s)L} (I - P^-)\mathcal{N}g(\cdot, s) ds, \end{aligned} \quad (5.8)$$

and the function $v = e^{\omega t}u$ belongs to $\mathbb{E}_1(\infty)$, with the estimate

$$\|v\|_{\mathbb{E}_1(\infty)} \leq C(\|u_0\|_{C^{2m+\alpha}(\overline{\Omega})} + \|e^{\omega t}f\|_{\mathbb{E}_0(\infty)} + \|e^{\omega t}g\|_{\mathbb{F}(\infty)}).$$

Proof. The proof follows [7, Thm. 0.1]. The novelty with respect to [7] is the appearance of systems of m boundary conditions (including possibly zero order boundary conditions) which will be treated with the method introduced in [20, Sect. 7].

Using the estimates (see [18, Prop. 2.3.3]) (which hold for small $\epsilon > 0$ and for $t > 0$)

$$\begin{aligned} \|P^- e^{tL}\|_{L(X)} &\leq C e^{-(\omega+\epsilon)t}, \\ \|LP^- e^{tL}\|_{L(X)} &\leq \frac{C e^{-(\omega+\epsilon)t}}{t}, \\ \|e^{-tL}(I - P^-)\|_{L(X)} &\leq C e^{-(\omega-\epsilon)t}, \end{aligned}$$

and arguing as in [18], it is easy to check that the function u given by (5.8) is bounded by $C e^{-\omega t}$.

Because of (5.3), we have $u = u_1 + u_2$, where u_1 is the function in the right hand side of (5.8) and

$$\begin{aligned} u_2(\cdot, t) &= e^{tL} \left((I - P^-)u_0 + \int_0^\infty e^{-sL}(I - P^-)(f(\cdot, s) + \mathcal{L}\mathcal{N}g(\cdot, s)) \, ds \right. \\ &\quad \left. - L \int_0^\infty e^{-sL}(I - P^-)\mathcal{N}g(\cdot, s) \, ds \right) \\ &= e^{tL}y, \quad t \geq 0. \end{aligned}$$

Due to the choice of ω the spectrum of $L + \omega I$ does not intersect the imaginary axis and the projection $(I - P^-)$ is the spectral projection associated to the unstable part of $\sigma(L + \omega I)$. And so because

$$e^{\omega t}u_2(\cdot, t) = e^{t(L+\omega I)}y,$$

y being an element of $(I - P^-)(X)$, we get $e^{\omega t}u_2(\cdot, t)$ is bounded in $[0, \infty)$ with values in X (which means that u is bounded) if and only if $y = 0$, i.e. if and only if (5.7) holds.

Let us prove that $v = e^{\omega t}u \in \mathbb{E}_1(\infty)$. Observe that v satisfies (5.1) with \mathcal{L} replaced by $\tilde{\mathcal{L}} = \mathcal{L} + \omega I$, and f and g replaced respectively by $\tilde{f} = f e^{\omega t}$ and $\tilde{g} = g e^{\omega t}$. In the following we shall set

$$\|\tilde{f}\| = \|\tilde{f}\|_{\mathbb{E}_0(\infty)}, \quad \|\tilde{g}\| = \|\tilde{g}\|_{\mathbb{F}(\infty)}.$$

Thanks to the compatibility condition (3.9) and to the regularity of the data, by Proposition 3.4, v belongs to $\mathbb{E}_1(1) = C^{2m+\alpha, 1+\frac{\alpha}{2m}}(\overline{\Omega} \times [0, 1])$ and

$$\|v\|_{\mathbb{E}_1(a)} \leq C(|u_0|_1 + \|\tilde{f}\| + \|\tilde{g}\|).$$

So we have to check that $v \in C^{2m+\alpha, 1+\frac{\alpha}{2m}}(\overline{\Omega} \times [1, \infty))$ and that its norm may be estimated in terms of the norms of the data.

As mentioned before due to the choice of ω the spectrum of $\tilde{L} = L + \omega I$ does not intersect the imaginary axis and the projection $(I - P^-)$ is the spectral projection associated to the unstable part of $\sigma(\tilde{L})$. Therefore the following estimates hold for some $\gamma > 0$:

$$\begin{aligned} \|\tilde{L}^k e^{t\tilde{L}} P^-\|_{L(X)} &\leq \frac{C_k e^{-\gamma t}}{t^k}, \\ \|\tilde{L}^k e^{-t\tilde{L}}(I - P^-)\|_{L(X)} &\leq C_k e^{-\gamma t}, \quad t > 0, \quad k \in \mathbb{N}. \end{aligned} \quad (5.9)$$

Let us define

$$\tilde{n}(t) := \mathcal{N}(\tilde{g}_1(t), \dots, \tilde{g}_m(t)) = \sum_{s=1}^m \mathcal{N}_s \mathcal{M}_s(\tilde{g}_1(t), \dots, \tilde{g}_s(t)), \quad (5.10)$$

$$\begin{cases} \tilde{n}_1(t) := \begin{cases} 0 & \text{if } m_1 > 0 \\ \mathcal{N}_1 \mathcal{M}_1(\tilde{g}_1) & \text{if } m_1 = 0, \end{cases} \\ \tilde{n}_2(t) := \tilde{n}(t) - \tilde{n}_1(t). \end{cases} \quad (5.11)$$

By decomposing v as $v = P^-v + (I - P^-)v$, using the equality (5.2) for the term P^-v , the equality (5.3) for the term $(I - P^-)v$ and taking into account (5.7), we could split $v(t) = v(\cdot, t)$ as $v = \sum_{i=1}^5 v_i$, where

$$\begin{aligned} v_1(t) &= e^{t\tilde{L}} P^-(u_0 - \tilde{n}(0)) + \int_0^t e^{(t-s)\tilde{L}} P^-[\tilde{f}(s) + \tilde{\mathcal{L}}\tilde{n}(s) - \tilde{n}'_1(0)] ds, \\ v_2(t) &= P^-\tilde{n}_1(t) - \int_0^t e^{(t-s)\tilde{L}} P^-(\tilde{n}'_1(s) - \tilde{n}'_1(0)) ds, \\ v_3(t) &= P^-\tilde{n}_2(0) - \tilde{L} \int_0^t e^{(t-s)\tilde{L}} P^-(\tilde{n}_2(s) - \tilde{n}_2(0)) ds, \\ v_4(t) &= - \int_t^\infty e^{(t-s)\tilde{L}} (I - P^-)[\tilde{f}(s) + \tilde{\mathcal{L}}\tilde{n}(s)] ds, \\ v_5(t) &= + \tilde{L} \int_t^\infty e^{(t-s)\tilde{L}} (I - P^-)\tilde{n}(s) ds. \end{aligned}$$

It is easy to see that v_1, v_2, v_3 satisfy the following equations:

$$\begin{cases} v'_1(t) = \tilde{\mathcal{L}}v_1(t) + P^-[\tilde{f}(t) + \tilde{\mathcal{L}}\tilde{n}(t) - \tilde{n}'_1(0)], & t \geq 0, \\ v_1(0) = P^-(u_0 + \tilde{n}(0)), \\ B_j v_1(t) = 0, \quad j = 1, \dots, m, & t \geq 0, \end{cases} \quad (5.12)$$

$v_2(t) = P^-\tilde{n}_1(t) + y(t)$, where $y(t)$ is a solution of

$$\begin{cases} y'(t) = \tilde{\mathcal{L}}y(t) - P^-[\tilde{n}'_1(t) - \tilde{n}'_1(0)], & t \geq 0, \\ y(0) = 0, \\ B_j y(t) = 0, \quad j = 1, \dots, m, & t \geq 0, \end{cases} \quad (5.13)$$

and $v_3(t) = P^-\tilde{n}_2(0) - \tilde{L}z(t)$, where $z(t)$ is a solution of

$$\begin{cases} z'(t) = \tilde{\mathcal{L}}z(t) + P^-[\tilde{n}_2(t) - \tilde{n}_2(0)], & t \geq 0, \\ z(0) = 0, \\ B_j z(t) = 0, & j = 1, \dots, m, \quad t \geq 0. \end{cases} \quad (5.14)$$

Furthermore, we need the following facts about the regularity of \tilde{n} , which are proven in [20], see (5),(11)-(13) of §7 in that paper.

Lemma 5.3.

$$\begin{cases} \tilde{n} & \in B([0, \infty); C^{2m+\alpha}(\overline{\Omega})) \cap C^{\frac{\alpha}{2m}}([0, \infty); C^{2m}(\overline{\Omega})), \\ \tilde{\mathcal{L}}\tilde{n} & \in B([0, \infty); C^\alpha(\overline{\Omega})) \cap C^{\frac{\alpha}{2m}}([0, \infty); X), \\ \tilde{n}_1 & \in B([0, \infty); C^{2m+\alpha}(\overline{\Omega})) \\ \tilde{\mathcal{L}}\tilde{n}_1 & \in B([0, \infty); C^\alpha(\overline{\Omega})) \\ \tilde{n}'_1 & \in C^{\frac{\alpha}{2m}}([0, \infty); X) \cap B([0, \infty); C^\alpha(\overline{\Omega})) \end{cases}$$

Let us first consider v_1 . Since $t \rightarrow \tilde{f}(\cdot, t)$, $t \rightarrow \tilde{\mathcal{L}}\tilde{n}(s)$ and $\tilde{n}'_1(0)$ belong to $C^{\frac{\alpha}{2m}}([0, \infty); X)$ by [18, Prop. 4.4.1(ii)] we have

$$\begin{cases} v_1 \in C^{1+\frac{\alpha}{2m}}([1, \infty); X), \\ v_1(t) \in \bigcap_{p>1} W^{2m,p}(\Omega), \\ v'_1 \in B([1, \infty); C^\alpha(\overline{\Omega})), \end{cases}$$

where we have used the fact that $D_{\tilde{L}}(\frac{\alpha}{2m}, \infty) \simeq C^\alpha(\overline{\Omega})$ (by Theorem 2.4 (ii)). On the other hand, since \tilde{f} , $\tilde{\mathcal{L}}\tilde{n}$, $\tilde{n}'_1(0)$ and v'_1 belong to $B([1, \infty); C^\alpha(\overline{\Omega}))$, by (5.12) we conclude that $\tilde{\mathcal{L}}v_1 \in B([1, \infty); C^\alpha(\overline{\Omega}))$.

Summing up we obtain

$$\begin{cases} v_1 \in C^{1+\frac{\alpha}{2m}}([1, \infty); X), & v'_1 \in B([1, \infty); C^\alpha(\overline{\Omega})), \\ \tilde{\mathcal{L}}v_1 \in B([1, \infty); C^\alpha(\overline{\Omega})), & v_1(t) \in \bigcap_{p>1} W^{2m,p}(\Omega), \quad t \in [1, \infty). \end{cases} \quad (5.15)$$

Considering v_2 , Since $n'_1(t) - n'_1(0) \in C^{\frac{\alpha}{2m}}([0, \infty); X)$ and $\tilde{\mathcal{L}}\tilde{n}_1(t), n'_1(t) \in B([1, \infty); C^\alpha(\overline{\Omega}))$, by [18, Prop. 4.4.1(ii)] and (5.13) we obtain similarly that v_2 satisfies the same properties as v_1 (see (5.15)).

Let us consider v_3 . we set for each

$$\begin{cases} s = 1, \dots, m & \text{if } m_1 = 0, \\ s = 2, \dots, m & \text{if } m_1 > 0, \end{cases}$$

$$\psi_s(t) = P^-\mathcal{N}_s\mathcal{M}_s(\tilde{g}_1(t) - \tilde{g}_1(0), \dots, \tilde{g}_s(t) - \tilde{g}_s(0)), \quad t \in [0, \infty),$$

and we first solve the problem

$$\begin{cases} v'_{3s}(t) = \tilde{L} v_{3s}(t) + \psi_s(t), & t \geq 0, \\ v_{3s}(0) = 0. \end{cases} \quad (5.16)$$

We have

$$\psi_s \in C^{\frac{2m+\alpha-m_s}{2m}}([0, \infty); D_{\tilde{L}}(\frac{m_s}{2m}, \infty)) \quad (5.17)$$

particularly because of the fact that $B_j \mathcal{N}_s = 0$ for $j < s$ (See (32) of §7 in [20] for more details).

Applying [18, Thm. 4.3.16] with $\theta = \frac{2m+\alpha-m_s}{2m}$, $\beta = \frac{m_s}{2m}$, we get for every $T > 0$

$$\tilde{L} v_{3s} \in C^{1+\frac{\alpha}{2m}}([0, T]; X), \quad v'_{3s} \in B([0, T]; D_{\tilde{L}}(1 + \frac{\alpha}{2m}, \infty)).$$

Looking at the proof of Theorem 4.3.16 in [18] and of the previous Theorem 4.3.1(iii) one sees that

$$\|\tilde{L} v_{3s}\|_{C^{1+\frac{\alpha}{2m}}([0, T]; X)} + \|\tilde{L} v'_{3s}\|_{B([0, T]; D_{\tilde{L}}(\frac{\alpha}{2m}, \infty))} \leq C \|\psi_s\|_{C^{\frac{2m+\alpha-m_s}{2m}}([0, \infty); C^{m_s}(\overline{\Omega}))},$$

with the constant C independent of T and so by

$$v_3(t) = P^- n_2(0) - \tilde{L} \sum_{s=1 \text{ or } 2}^m v_{3s}(t), \quad (5.18)$$

we get $v'_3 \in B([1, \infty); C^\alpha(\overline{\Omega})) \cap C^{\frac{\alpha}{2m}}([1, \infty); X)$. Moreover by (5.14) we easily see that

$$v'_3 = \tilde{\mathcal{L}} v_3 - \tilde{\mathcal{L}} P^- n_2(t)$$

and so $\tilde{\mathcal{L}} v_3 \in B([1, \infty); C^\alpha(\overline{\Omega}))$. Summing up we obtain that v_3 satisfies the same properties as v_1 (see (5.15)).

Considering v_4 , Since again $t \rightarrow \tilde{f}(\cdot, t)$, $t \rightarrow \tilde{\mathcal{L}} \tilde{n}(s)$ belong to $C^{\frac{\alpha}{2m}}([0, \infty); X)$, by [18, Prop. 4.4.2(ii)] we easily see that it satisfies the same properties as v_1 (see (5.15)).

Finally let us consider v_5 . By estimates (5.9), it is obviously bounded with values in $D(L^k)$ for every $k \in \mathbb{N}$. Moreover $v'_5 = \tilde{L} v_5 - \tilde{L}(I - P^-) \tilde{n}$ is Hölder continuous with value in X and is bounded with value in $C^\alpha(\overline{\Omega})$. the latter comes from the fact that $\tilde{L}(I - P^-) \tilde{n} \in B([1, \infty); C^\alpha(\overline{\Omega}))$. And so v_3 satisfies the same properties as v_1 (see (5.15)).

Since $v = \sum_{i=1}^5 v_i$, we have

$$\begin{cases} v \in C^{1+\frac{\alpha}{2m}}([1, \infty); X), & v' \in B([1, \infty); C^\alpha(\overline{\Omega})), \\ \tilde{\mathcal{L}} v \in B([1, \infty); C^\alpha(\overline{\Omega})), & v(t) \in \bigcap_{p>1} W^{2m,p}(\Omega), \quad t \in [1, \infty). \end{cases} \quad (5.19)$$

Now we only need to show that

$$v \in B([1, \infty); C^{2m+\alpha}(\overline{\Omega}))$$

and this can be done by using (iv) of Theorem 2.4, by virtue of (5.19) and the fact that $B_j v = \tilde{g}_j \in B([1, \infty); C^{2m+\alpha-m_j}(\partial\Omega))$.

It follows that $v \in C^{2m+\alpha, 1+\frac{\alpha}{2m}}(\overline{\Omega} \times [1, \infty))$, and

$$\|v\|_{C^{2m+\alpha, 1+\frac{\alpha}{2m}}(\overline{\Omega} \times [1, \infty))} \leq C(\|u_0\|_X + \|\tilde{f}\| + \|\tilde{g}\|),$$

which finishes the proof. \square

5.2 An extension operator and asymptotic behavior in linear systems

In order to apply the semigroup theory, similarly as previous section, to obtain results for the asymptotic behavior of linear systems, we need to construct explicitly an extension operator for the case of vector-valued unknowns.

5.2.1 An extension operator

Let us recall our linear boundary problem:

$$(B_j u)(x) = \sum_{|\beta| \leq m_j} b_\beta^j(x) \nabla^\beta u(x), \quad x \in \partial\Omega, \quad j = 1, \dots, mN. \quad (5.20)$$

Here $u = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}$, b_β^j are N -dimensional row-vectors and

$$0 \leq m_1 \leq m_2 \leq \dots \leq m_{mN} \leq 2m - 1.$$

Recall that $n_j \geq 0$ are the number of j th order boundary conditions for $j = 0, \dots, 2m - 1$.

Our goal is to construct explicitly a linear and bounded operator E such that for all $\theta' \in [0, \alpha]$,

$$\begin{cases} g_j \in C^{2m+\theta'-m_j}(\partial\Omega), \quad j = 1, \dots, mN \implies E(g_1, \dots, g_{mN}) \in C^{2m+\theta'}(\overline{\Omega}), \\ B_j E(g_1, \dots, g_{mN}) = g_j, \quad j = 1, \dots, mN. \end{cases} \quad (5.21)$$

Note that as you have already seen such a lifting operator were constructed explicitly in [20, Thm. 6.3], i.e. Theorem 5.1 for systems of m boundary conditions for a single unknown, i.e. $N = 1$.

The strategy for proving the existence of the extension operator E satisfying (5.21) is as follows: At first, by using the normality condition (2.8), we will reduced our linear system to an uncoupled linear system and then with the help

of the scalar result, i.e. Theorem 5.1, applying it to each component, we finish the proof.

In the following, we set γ_j for the j th order normal derivatives precisely, for $j = 0, \dots, 2m - 1$

$$\gamma_j u := D^j u \left[\overbrace{\nu, \dots, \nu}^{j\text{-times}} \right] |_{\partial\Omega},$$

which should be understood component-wise. Recall that $\nu(x)$ is the unit outer normal to $\partial\Omega$ at the point x .

Theorem 5.4. *Assume the operators B_j satisfy the regularity condition (R_2) and normality condition (2.8). Then there exist a linear bounded operator E satisfying (5.21).*

Proof. For notational convenience, without loss of generality we could assume that $n_k \neq 0$ for all k between 0 and $2m - 1$, i.e. we have here included all orders k between 0 and $2m - 1$. Indeed, if $n_k = 0$ for some k , we could simply add the boundary conditions $\gamma_k u = 0$.

Setting $u = E(g_1, \dots, g_{mN})$ in (5.21) and decomposing derivatives into normal and tangential derivatives, the last condition in (5.21) can be rewritten as

$$\begin{cases} S_{00}(x)\gamma_0 u = \varphi_0, \\ S_{11}(x)\gamma_1 u + \text{tangential derivatives} + \text{zero order normal derivatives} = \varphi_1, \\ \vdots \\ S_{2m-1, 2m-1}(x)\gamma_{2m-1} u + \text{tangential derivatives} \\ \quad + \text{lower order normal derivatives} = \varphi_{2m-1}. \end{cases} \quad (5.22)$$

Here, for all $k = 0, \dots, 2m - 1$

$$S_{kk}(x) := \begin{pmatrix} \sum_{|\beta|=k} b_{\beta}^{j_1}(x)(\nu(x))^{\beta} \\ \vdots \\ \sum_{|\beta|=k} b_{\beta}^{j_{n_k}}(x)(\nu(x))^{\beta} \end{pmatrix}_{n_k \times N}$$

where j_i is such that $m_{j_i} = k$ for all $1 \leq i \leq n_k$ and

$$\varphi_0 := \begin{pmatrix} g_1 \\ \vdots \\ g_{n_0} \end{pmatrix}_{n_0 \times 1}, \quad \varphi_{k+1} := \begin{pmatrix} g_{\sum_{i=0}^k n_i + 1} \\ \vdots \\ g_{\sum_{i=0}^{k+1} n_i} \end{pmatrix}_{n_{k+1} \times 1}.$$

By normality condition, S_{kk} are surjective and therefore there exist matrices R_{kk} which have the same regularity as S_{kk} such that

$$S_{kk} R_{kk} = I. \quad (5.23)$$

Now we are in a position to reduce the linear system (5.22) to an uncoupled linear system. First of all by (5.23), it is enough to consider an uncoupled zero order normal boundary conditions

$$\gamma_0 u = R_{00} \varphi_0 \quad (5.24)$$

instead of the first condition in (5.22).

Using (5.24), all tangential derivatives and of course all zero order normal derivatives can be calculated and therefore the second condition in (5.22) can be rewritten as

$$S_{11}(x) \gamma_1 u = \varphi_1(x) + \eta_1(x),$$

for some $\eta_1(x)$ which can be calculated in terms of $R_{00} \varphi_0$ or precisely in terms of (g_1, \dots, g_{n_0}) .

Now again by (5.23), it is enough to consider an uncoupled first order normal boundary conditions $\gamma_1 u = R_{11}(\varphi_1 + \eta_1)$ and proceed further and so by iteration, we are led to the following uncoupled linear system of normal boundary conditions:

$$\begin{cases} \gamma_0 u = \psi_0, \\ \gamma_1 u = \psi_1, \\ \vdots \\ \gamma_{2m-1} u = \psi_{2m-1}, \end{cases} \quad (5.25)$$

where $\psi_0 = R_{00} \varphi_0$ and $\psi_k = R_{kk}(\varphi_k + \eta_k)$, for some η_k which can be calculated in terms of $\psi_0, \dots, \psi_{k-1}$. It is easy to see that

$$\psi_k(x) = \begin{pmatrix} \psi_{k1}(x) \\ \vdots \\ \psi_{kN}(x) \end{pmatrix} \in C^{2m+\alpha-k}(\partial\Omega).$$

By Theorem 5.1, applying it to each of u_i , for the boundary operators $B_j = \gamma_{j-1}$, we could define a linear and bounded operator E as follows:

$$E(g_1, \dots, g_{mN}) := \left(\mathcal{N}(\psi_{01}, \psi_{11}, \dots, \psi_{2m-1,1}), \dots, \mathcal{N}(\psi_{0N}, \psi_{1N}, \dots, \psi_{2m-1,N}) \right), \quad (5.26)$$

where the operator

$$\mathcal{N}(\psi_{0i}, \psi_{1i}, \dots, \psi_{2m-1,i}) = \sum_{s=1}^{2m} \mathcal{N}_s \mathcal{M}_s(\psi_{0i}, \dots, \psi_{s-1,i})$$

is the extension operator given in Theorem 5.1 for the boundary operators $B_j = \gamma_{j-1}$, for $j = 1, \dots, 2m$.

Note that by looking at the proof of Theorem 5.1 or equivalently Theorem 6.3 in [20], one sees that the number of boundary conditions in Theorem 6.3 in

[20] can be replaced by any m' as far as the normality condition is satisfied and $m_j \leq 2m - 1$ for all $j = 1, \dots, m'$, which is a case in our situation.

Moreover, by (5.6) for each $v \in C(\partial\Omega)$

$$(B_j(\mathcal{N}_s v_1, \dots, \mathcal{N}_s v_N))(x) \equiv 0, \quad x \in \partial\Omega, \quad m_j < s - 1. \quad (5.27)$$

And finally the regularity condition in (5.21) comes from the fact that the operator \mathcal{N} has a similar regularity property and this finishes the proof. \square

5.2.2 Asymptotic behavior in linear systems

Here we extend the result of section 5.1 to the systems of mN boundary conditions for a linear system. Precisely we consider the linear problem (3.8), i.e.

$$\begin{cases} \partial_t u + Au = f(t) & \text{in } \Omega, \quad t \geq 0, \\ Bu = g(t) & \text{on } \partial\Omega, \quad t \geq 0, \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \quad (5.28)$$

where $u = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}$, Ω is a bounded open set in \mathbb{R}^n with $C^{2m+\alpha}$ boundary,

$0 < \alpha < 1$, $g = (g_1, \dots, g_{mN})$, $B = (B_1, \dots, B_{mN})$, $u_0 \in C^{2m+\alpha}(\overline{\Omega})$ and the operators A and B satisfy the (R_2) , (L-S), (SP) and normality condition (2.8).

The realisation $-A_0$ of $-A$ with homogeneous boundary conditions in $C(\overline{\Omega})$, defined in (2.5), is a sectorial operator thanks to Theorem 2.4(i).

Similarly as before, we set $\mathcal{L} = -A$ and $L = -A_0$. Moreover if f, g and u_0 are regular enough the unique solution of (5.28) is given by the extension of the Balakrishnan formula with some adaptations.

Note that by our explicit construction of the extension operator (see (5.26)), we simply can extend Theorem 4.1 in [20] to cover the linear systems using the same technique and therefore we get the following formula.

$$\begin{aligned} u(\cdot, t) &= e^{tL}(u_0 - n(0)) + \int_0^t e^{(t-s)L}[f(\cdot, s) + \mathcal{L}n(s) - n'_1(0)]ds \\ &\quad + n_1(t) - \int_0^t e^{(t-s)L}(n'_1(s) - n'_1(0))ds \\ &\quad - L \int_0^t e^{(t-s)L}[n_2(s) - n_2(0)]ds + n_2(0) \\ &= e^{tL}u_0 + \int_0^t e^{(t-s)L}[f(\cdot, s) + \mathcal{L}n(s)]ds \\ &\quad - L \int_0^t e^{(t-s)L}n(s)ds, \quad 0 \leq t \leq T. \end{aligned} \quad (5.29)$$

Here

$$n(t) = E(g_1(t), \dots, g_{mN}(t))$$

and similarly as before

$$\begin{cases} n_1(t) = \begin{cases} 0 & \text{if } m_1 > 0 \\ (\mathcal{N}_1 \mathcal{M}_1(\psi_{0,1}), \dots, \mathcal{N}_1 \mathcal{M}_1(\psi_{0,N})) & \text{if } m_1 = 0, \end{cases} \\ n_2(t) = n(t) - n_1(t), \end{cases}$$

where $\psi_0 = (\psi_{0,1}, \dots, \psi_{0,N})^T = R_{00}\varphi_0$ which can be written in terms of g_1, \dots, g_{n_0} .

Theorem 5.5. *Let $0 < \omega < -\max\{\operatorname{Re}\lambda : \lambda \in \sigma^-(-A_0)\}$. Let f be such that $(\sigma, t) \rightarrow e^{\omega t}f(\sigma, t) \in \mathbb{E}_0(\infty)$, let g be such that $(\sigma, t) \rightarrow e^{\omega t}g(\sigma, t) \in \mathbb{F}(\infty)$ and let $u_0 \in C^{2m+\alpha}(\overline{\Omega})$ satisfy the compatibility condition (3.9). Then $v(\sigma, t) = e^{\omega t}u(\sigma, t)$ is bounded in $[0, +\infty) \times \overline{\Omega}$ if and only if*

$$\begin{aligned} (I - P^-)u_0 = & - \int_0^{+\infty} e^{-sL}(I - P^-)[f(\cdot, s) + \mathcal{L}Eg(\cdot, s)] ds \\ & + L \int_0^\infty e^{-sL}(I - P^-)Eg(\cdot, s) ds. \end{aligned} \quad (5.30)$$

In this case, u is given by

$$\begin{aligned} u(\cdot, t) = & e^{tL}P^-u_0 + \int_0^t e^{(t-s)L}P^-[f(\cdot, s) + \mathcal{L}Eg(\cdot, s)] ds \\ & - L \int_0^t e^{(t-s)L}P^-Eg(\cdot, s) ds \\ & - \int_t^{+\infty} e^{(t-s)L}(I - P^-)[f(\cdot, s) + \mathcal{L}Eg(\cdot, s)] ds \\ & + L \int_t^{+\infty} e^{(t-s)L}(I - P^-)Eg(\cdot, s) ds, \end{aligned} \quad (5.31)$$

and the function $v = e^{\omega t}u$ belongs to $\mathbb{E}_1(\infty)$, with the estimate

$$\|v\|_{\mathbb{E}_1(\infty)} \leq C(\|u_0\|_{C^{2m+\alpha}(\overline{\Omega})} + \|e^{\omega t}f\|_{\mathbb{E}_0(\infty)} + \|e^{\omega t}g\|_{\mathbb{F}(\infty)}).$$

Proof. The proof is exactly the same as the proof of Theorem 5.2. More precisely, as you have seen, we used the abstract theories, i.e. the theory of semi-groups of linear operators in the proof, except the part related to the function v_3 . Due to our explicit construction of the extension operator (see (5.26)) and taking into account (5.6) (in order to obtain the same result as (5.17)), we could work component-wise and get the same estimate for the function v_3 . This finishes the proof. \square

Theorem 5.5 has an important corollary in the stable case when $\sigma(-A_0) = \sigma^-(-A_0)$.

Corollary 5.6. *Assume that $\omega_A := \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(-A_0)\} < 0$ and fix $\omega \in (0, -\omega_A)$. Let f be such that $(\sigma, t) \rightarrow e^{\omega t} f(\sigma, t) \in \mathbb{E}_0(\infty)$, let g be such that $(\sigma, t) \rightarrow e^{\omega t} g(\sigma, t) \in \mathbb{F}(\infty)$ and let $u_0 \in C^{2m+\alpha}(\overline{\Omega})$ satisfy the compatibility condition (3.9). Then $v(\sigma, t) = e^{\omega t} u(\sigma, t)$ belongs to $\mathbb{E}_1(\infty)$ and*

$$\|v\|_{\mathbb{E}_1(\infty)} \leq C(\|u_0\|_{C^{2m+\alpha}(\overline{\Omega})} + \|e^{\omega t} f\|_{\mathbb{E}_0(\infty)} + \|e^{\omega t} g\|_{\mathbb{F}(\infty)}).$$

5.3 Proof of Proposition 3.7

In fact we are following the steps in the proof of Theorem 4.1 in [19].

For $0 \leq t \leq a$,

$$\begin{aligned} \|e^{\sigma t} F(z_1(t, \cdot)) - e^{\sigma t} F(z_2(t, \cdot))\|_{C^\alpha(\overline{\Omega})} &\leq K(r) \|e^{\sigma t} (z_1(t, \cdot) - z_2(t, \cdot))\|_{C^{2m+\alpha}(\overline{\Omega})} \\ &\leq K(r) \|e^{\sigma t} (z_1 - z_2)\|_{\mathbb{E}_1(a)}, \end{aligned}$$

$$\begin{aligned} \|e^{\sigma t} G_j(z_1(t, \cdot)) - e^{\sigma t} G_j(z_2(t, \cdot))\|_{C^{2m+\alpha-m_j}(\partial\Omega)} &\leq H_j(r) \|e^{\sigma t} (z_1(t, \cdot) - z_2(t, \cdot))\|_{C^{2m+\alpha}(\overline{\Omega})} \\ &\leq H_j(r) \|e^{\sigma t} (z_1 - z_2)\|_{\mathbb{E}_1(a)}, \end{aligned}$$

while for $0 \leq s \leq t \leq a$,

$$\begin{aligned} &\|e^{\sigma t} F(z_1(t, \cdot)) - e^{\sigma t} F(z_2(t, \cdot)) - e^{\sigma s} F(z_1(s, \cdot)) + e^{\sigma s} F(z_2(s, \cdot))\|_{C(\overline{\Omega})} \\ &= \left\| \int_0^1 e^{\sigma t} F'(\lambda z_1(t, \cdot) + (1-\lambda)z_2(t, \cdot)) (z_1(t, \cdot) - z_2(t, \cdot)) \right. \\ &\quad \left. - e^{\sigma s} F'(\lambda z_1(s, \cdot) + (1-\lambda)z_2(s, \cdot)) (z_1(s, \cdot) - z_2(s, \cdot)) d\lambda \right\|_{C(\overline{\Omega})} \\ &\leq \int_0^1 \left\| F'(\lambda z_1(t, \cdot) + (1-\lambda)z_2(t, \cdot)) - F'(\lambda z_1(s, \cdot) + (1-\lambda)z_2(s, \cdot)) \right\| \\ &\quad \times e^{\sigma t} \|z_1(t, \cdot) - z_2(t, \cdot)\|_{C(\overline{\Omega})} d\lambda \\ &\quad + \int_0^1 \|F'(\lambda z_1(s, \cdot) + (1-\lambda)z_2(s, \cdot))\| (e^{\sigma t} \|z_1(t, \cdot) - z_2(t, \cdot)\| - e^{\sigma s} \|z_1(s, \cdot) - z_2(s, \cdot)\|) d\lambda \\ &\leq \frac{L}{2} (\|z_1(t, \cdot) - z_1(s, \cdot)\|_{C^{2m}(\overline{\Omega})} + \|z_2(t, \cdot) - z_2(s, \cdot)\|_{C^{2m}(\overline{\Omega})}) e^{\sigma t} \|z_1(t, \cdot) - z_2(t, \cdot)\|_{C^{2m}(\overline{\Omega})} \\ &\quad + Lr \|e^{\sigma t} (z_1(t, \cdot) - z_2(t, \cdot)) - e^{\sigma s} (z_1(s, \cdot) - z_2(s, \cdot))\|_{C^{2m}(\overline{\Omega})} \\ &\leq \frac{L}{2} (t-s)^{\frac{\alpha}{2m}} (\|z_1\|_{C^{\frac{\alpha}{2m}}((0,a), C^{2m}(\overline{\Omega}))} + \|z_2\|_{C^{\frac{\alpha}{2m}}((0,a), C^{2m}(\overline{\Omega}))}) \|e^{\sigma t} (z_1 - z_2)\|_{\mathbb{E}_1(a)} \\ &\quad + Lr (t-s)^{\frac{\alpha}{2m}} \|e^{\sigma t} (z_1 - z_2)\|_{C^{\frac{\alpha}{2m}}((0,a), C^{2m}(\overline{\Omega}))} \\ &\leq 2Lr (t-s)^{\frac{\alpha}{2m}} \|e^{\sigma t} (z_1 - z_2)\|_{\mathbb{E}_1(a)}, \end{aligned}$$

the last inequality being a consequence of Lemma 3.3 and the fact that $z_1, z_2 \in B_{\mathbb{E}_1(a)}(0, r)$.

Since $1 + \frac{\alpha}{2m} - \frac{m_j}{2m} < 1$ for j with $m_j \geq 1$, Arguing similarly we get

$$\begin{aligned} & \|e^{\sigma t} G_j(z_1(t, \cdot)) - e^{\sigma t} G_j(z_2(t, \cdot)) - e^{\sigma s} G_j(z_1(s, \cdot)) + e^{\sigma s} G_j(z_2(s, \cdot))\|_{C(\partial\Omega)} \\ & \leq 2Lr(t-s)^{1+\frac{\alpha}{2m}-\frac{m_j}{2m}} \|e^{\sigma t}(z_1 - z_2)\|_{\mathbb{E}_1(a)}, \end{aligned}$$

where we have used the following embedding

$$\mathbb{E}_1(a) \hookrightarrow C^{1+\frac{\alpha}{2m}-\frac{m_j}{2m}}((0, a), C^{m_j}(\overline{\Omega})),$$

which is a consequence of Lemma 3.3.

For j such that $m_j = 0$, we have to estimate the complete norm, which includes the time derivative, i.e. $\|e^{\sigma t}(G_j(z_1) - G_j(z_2))\|_{C^{1+\frac{\alpha}{2m}}(I, C(\partial\Omega))}$. The proof is again similar, but for the convenience we give some details of the main part of it namely, estimating $\|e^{\sigma t} \frac{d}{dt}(G_j(z_1) - G_j(z_2))\|_{C^{\frac{\alpha}{2m}}(I, C(\partial\Omega))}$.

For $0 \leq s \leq t \leq a$, we have

$$\begin{aligned} & \|e^{\sigma t} G'_j(z_1(t, \cdot)) z'_1(t, \cdot) - e^{\sigma t} G'_j(z_2(t, \cdot)) z'_2(t, \cdot) - e^{\sigma s} G'_j(z_1(s, \cdot)) z'_1(s, \cdot) + e^{\sigma s} G'_j(z_2(s, \cdot)) z'_2(s, \cdot)\|_{C(\partial\Omega)} \\ & \leq \left\| \int_0^1 e^{\sigma t} G''_j \left(\lambda z_1(t, \cdot) + (1-\lambda) z_2(t, \cdot) \right) \left(z_1(t, \cdot) - z_2(t, \cdot) \right) \left(\lambda z'_1(t, \cdot) + (1-\lambda) z'_2(t, \cdot) \right) \right. \\ & \quad \left. - e^{\sigma s} G''_j \left(\lambda z_1(s, \cdot) + (1-\lambda) z_2(s, \cdot) \right) \left(z_1(s, \cdot) - z_2(s, \cdot) \right) \left(\lambda z'_1(s, \cdot) + (1-\lambda) z'_2(s, \cdot) \right) d\lambda \right\|_{C(\partial\Omega)} \\ & \quad + \left\| \int_0^1 e^{\sigma t} G'_j \left(\lambda z_1(t, \cdot) + (1-\lambda) z_2(t, \cdot) \right) \left(z'_1(t, \cdot) - z'_2(t, \cdot) \right) \right. \\ & \quad \left. - e^{\sigma s} G'_j \left(\lambda z_1(s, \cdot) + (1-\lambda) z_2(s, \cdot) \right) \left(z'_1(s, \cdot) - z'_2(s, \cdot) \right) d\sigma \right\|_{C(\partial\Omega)} \\ & \leq \int_0^1 \left\| \left(G''_j \left(\lambda z_1(t, \cdot) + (1-\lambda) z_2(t, \cdot) \right) - G''_j \left(\lambda z_1(s, \cdot) + (1-\lambda) z_2(s, \cdot) \right) \right) \right. \\ & \quad \left. \times e^{\sigma t} \left(z_1(t, \cdot) - z_2(t, \cdot) \right) \left(\lambda z'_1(t, \cdot) + (1-\lambda) z'_2(t, \cdot) \right) \right\|_{C(\partial\Omega)} d\lambda \\ & \quad + \int_0^1 \left\| G''_j \left(\lambda z_1(s, \cdot) + (1-\lambda) z_2(s, \cdot) \right) \left(e^{\sigma t} \left(z_1(t, \cdot) - z_2(t, \cdot) \right) - e^{\sigma s} \left(z_1(s, \cdot) - z_2(s, \cdot) \right) \right) \right. \\ & \quad \left. \times \left(\lambda z'_1(t, \cdot) + (1-\lambda) z'_2(t, \cdot) \right) \right\|_{C(\partial\Omega)} d\lambda \\ & \quad + \int_0^1 \left\| G''_j \left(\lambda z_1(s, \cdot) + (1-\lambda) z_2(s, \cdot) \right) e^{\sigma s} \left(z_1(s, \cdot) - z_2(s, \cdot) \right) \right. \\ & \quad \left. \times \left(\lambda z'_1(t, \cdot) + (1-\lambda) z'_2(t, \cdot) - \lambda z'_1(s, \cdot) - (1-\lambda) z'_2(s, \cdot) \right) \right\|_{C(\partial\Omega)} d\lambda \\ & \quad + \int_0^1 \left\| \left(G'_j \left(\lambda z_1(t, \cdot) + (1-\lambda) z_2(t, \cdot) \right) - G'_j \left(\lambda z_1(s, \cdot) + (1-\lambda) z_2(s, \cdot) \right) \right) \right. \\ & \quad \left. \times e^{\sigma t} \left(z'_1(t, \cdot) - z'_2(t, \cdot) \right) \right\|_{C(\partial\Omega)} d\lambda \\ & \quad + \int_0^1 \left\| G'_j \left(\lambda z_1(s, \cdot) + (1-\lambda) z_2(s, \cdot) \right) \left(e^{\sigma t} \left(z'_1(t, \cdot) - z'_2(t, \cdot) \right) - e^{\sigma s} \left(z'_1(s, \cdot) - z'_2(s, \cdot) \right) \right) \right\|_{C(\partial\Omega)} d\lambda \\ & \leq (12Lr^2 + 2Lr) (t-s)^{\frac{\alpha}{2m}} \|e^{\sigma t}(z_1 - z_2)\|_{\mathbb{E}_1(a)}, \end{aligned}$$

where we have used the fact that $\lambda, 1 - \lambda \leq 1$ and

$$\mathbb{E}_1(a) \hookrightarrow C^{\frac{\alpha}{2m}}(I, C^{2m}(\overline{\Omega})) \hookrightarrow C^{\frac{\alpha}{2m}}(I, C(\overline{\Omega})).$$

Summing up, by adding all constants we get a constant $D(r)$ such that $D(r)$ goes to zero as $r \rightarrow 0$ and so the statement follows.

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